Math 474 HW #4

Due 2:00 PM Friday, Oct. 30

1. (Required Problem) Find the first fundamental form of the following surfaces:
   (a) The ellipsoid $\vec{x}(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$.
   (b) The elliptic paraboloid $\vec{x}(u,v) = (au\cos v, bu\sin v, u^2)$.
   (c) The hyperbolic paraboloid $\vec{x}(u,v) = (au\cosh v, bu\sinh v, u^2)$.
   (d) The two-sheeted hyperboloid $\vec{x}(u,v) = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$.

Choose an additional 4 of the following 7 problems to do.

2. Consider the sphere $x^2 + y^2 + z^2 = 1$ centered at the origin in $\mathbb{R}^3$. We can construct a very important map st from the sphere to the xy-plane by defining $\mathrm{st}(p)$ to be the intersection of the line through $p \in S^2$ and $(0,0,1)$ with the xy-plane (of course this map is not well defined at the point $(0,0,1)$).
   (a) Show that the inverse map $\mathrm{st}^{-1} : \mathbb{R}^2 \to S^2\backslash\{(0,0,1)\}$ is given by
   $$\mathrm{st}^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$
   (b) Show that st^{-1} is a regular parametrization of $S^2\backslash\{(0,0,1)\}$.
   (c) Find the first fundamental form I_p of this parametrization (as a matrix).
3. Let Σ be the hyperboloid of revolution given by $x^2+y^2-z^2 = 1$, which intersects the xy-plane in the unit circle $s^2 + y^2 = 1$. Prove that Σ intersects the xy-plane orthogonally.
4. (Shifrin Problem 2.1.6) A parametrization $\vec{x}(u,v)$ of a surface Σ is called *conformal* if angles measured in the uv-plane agree with the corresponding angles in $T_p\Sigma$ for all points $p \in \Sigma$. Prove that the parametrization $\vec{x}(u,v)$ is conformal if and only if $E = G$ and $F = 0$.
5. Suppose Σ is parametrized by $\vec{x} : \mathbb{R}^2 \to \Sigma$ where
   $$\vec{x}(u,v) = \alpha_1(u) + \alpha_2(v),$$
   where α_1 and α_2 are regular curves. For example, if $\alpha_1(u) = (\cos u, \sin u, 0)$ and $\alpha_2(v) = (0,0,v)$, then Σ is an infinite cylinder.

   Show that the tangent planes along the curve
   $$\beta(s) = \vec{x}(s, v_0)$$
   are parallel to a line. What's the line?
6. If $\alpha : [0,1] \to \mathbb{R}^3$ is a regular parametrized curve with unit tangent vector $\vec{T}(s)$, then the vectors $\vec{N}_1, \vec{N}_2 : [0,1] \to \mathbb{R}^3$ form a *framing* for α if the triple $(\vec{T}(s), \vec{N}_1(s), \vec{N}_2(s))$ is an orthonormal basis for $\mathbb{R}^3$ for all $s \in [0,1]$ (for example, the Frenet frame and the Bishop frame give framings).

   The *tube* around α of radius r is the surface parametrized by
   $$\vec{x}(u,v) = \alpha(u) + r\left(\cos v N_1(u) + \sin v N_2(u)\right).$$
   Find the Gauss map $\vec{N}(u,v)$ of the tube.

7. Let $\alpha(s) = (x(s), 0, z(s))$ be an arclength parametrized curve in the xz-plane which does not intersect the z-axis. The corresponding surface of revolution Σ generated by α is given by rotating α around the z-axis. This surface has the parametrization

$$\vec{x}(u, v) = (x(v)\cos u, x(v)\sin u, z(v)).$$

   (a) (Pappus' Theorem) Prove that the area of Σ is

$$\text{Area}(\Sigma) = 2\pi \int_0^L x(s)\,\mathrm{ds},$$

   where L is the length of α.

   (b) Suppose α is the circle of radius r_1 in the xz-plane centered at $(r_2, 0, 0)$ with $r_2 > r_1$. Then the corresponding surface of revolution is a torus; use (a) to compute its area.

8. (Shifrin Problem 2.1.14) A parametrization $\vec{x}(u, v)$ of a regular surface Σ is called a *Chebyshev net*[1] if the opposite sides of any quadrilateral formed by the coordinate curves $u = \text{const}$ and $v = \text{const}$ have equal length.

   (a) Prove that the parametrization is a Chebyshev net if and only if $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$.

   (b) Prove that it is possible to locally reparametrize a Chebyshev net by $\widetilde{\vec{x}}(\widetilde{u}, \widetilde{v})$ so that $\widetilde{E} = \widetilde{G} = 1$ and $\widetilde{F} = \cos\theta$ (meaning that the $\widetilde{u}$- and $\widetilde{v}$-curves are parametrized by arclength and meet at angle θ, which is really a function $\theta(\widetilde{u}, \widetilde{v})$ of $\widetilde{u}$ and $\widetilde{v}$).

[1] "Chebyshev" is just one of many Roman transliterations of the Russian name Чебышёв. This is the same name as Chebychev, Chebysheff, Tchebyshef, Tchebychev, Tschebyschev, etc.