Math 474 HW #3

Due 2:00 PM Friday, Oct. 2

Choose 3 of the following 5 problems to write up and turn in.

1. (Shifrin Problem 1.3.1) Prove that the shortest path between two points $\vec{p}$ and $\vec{q}$ on the sphere is an arc of a great circle connecting them, and show that the length of this path is $\arccos(\vec{p}\cdot\vec{q})$. (See the statement of the problem in Shifrin for some hints on how to set things up.)

2. (Shifrin Problem 1.3.4) Suppose $\alpha(s)$ is a simple closed plane curve with $0 < \kappa(s) \leq c$ where c is some constant. Prove that $\text{length}(\alpha) \geq 2\pi/c$.

3. Let $\alpha(s)$ be a closed convex plane curve. The *parallel curve* to α at distance r is the curve

$$\beta(s) = \alpha(s) - rN(s),$$

where $N(s)$ is the unit normal to α. Let $\kappa_\alpha(s)$ be the curvature of $\alpha(s)$ and let $\kappa_\beta(s)$ be the curvature of $\beta(s)$. Prove that

   (a) $\text{length}(\beta) = \text{length}(\alpha) + 2\pi r$.

   (b) $\text{Area}(\beta) = \text{Area}(\alpha) + r\,\text{length}(\alpha) + \pi r^2$.

   (c) $\kappa_\beta(s) = \frac{\kappa_\alpha(s)}{1+r\kappa_\alpha(s)}$ (Note: β is probably *not* parametrized by arclength)

4. For a *piecewise-smooth* closed curve in $\mathbb{R}^3$, we can define the total curvature of the curve to be

$$\int \kappa(s)ds + \sum_{i=1}^{n} \theta_i,$$

where the θ_i are the exterior angles at the corners of the curve (see Shifrin's problem 1.3.12 for a precise definition of the exterior angle and a picture).

Generalize Fenchel's Theorem to show that the total curvature of a piecewise-smooth curve is $\geq 2\pi$, with equality if and only if the curve is planar and convex.

5. (Milnor's integral geometric total curvature formula) Suppose $\alpha(s)$ is a curve in $\mathbb{R}^3$ with curvature $\kappa(s)$. For any $\vec{p} \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$), let $\alpha_{\vec{p}}(s)$ be the projection of α to the line determined by $\vec{p}$. This is not a regular curve, but we can define its total curvature as

$$\kappa_{\vec{p}} := \pi \cdot (\text{total number of times } \alpha_{\vec{p}} \text{ changes direction}).$$

Prove that

$$\int \kappa(s)ds = \frac{1}{4\pi}\int_{\vec{p}\in S^2} \kappa_{\vec{p}}\,d\vec{p}.$$