Math 474 HW #2

Due 2:00 PM Friday, Sept. 25

Choose 5 of the following 8 problems to write up and turn in.

1. Suppose $a \neq b$ and consider the ellipse $\alpha(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$. Compute the curvature of the ellipse and show that it has exactly four vertices.

2. (Shifrin Problem 1.2.7) Suppose $\alpha(s)$ is parametrized by arclength and has the property that $\|\alpha(s)\| \leq \|\alpha(s_0)\| = R$ for all s sufficiently close to s_0. Show that $\kappa(s_0) \geq \frac{1}{R}$. (Hint: consider the function $f(s) = \|\alpha(s)\|^2$. What do you know about $f''(s_0)$?)

3. (Shifrin Problem 1.2.8) Let α be a regular curve parametrized by arclength with nonvanishing curvature. The *normal line* to α at $\alpha(s)$ is the line through $\alpha(s)$ with direction vector $N(s)$ (the Frenet normal). Suppose all normal lines to α pass through a given fixed point. What can you say about the curve α? Does your answer change if the curve is not regular?

4. Let α be a regular curve parametrized by arclength. The *tangent line* to α at $\alpha(s)$ is the line through $\alpha(s)$ with direction vector $T(s)$ (the unit tangent vector). Suppose all tangent lines to α pass through a given fixed point. What can you say about the curve α? Does your answer change if the curve is not regular?

5. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve with nonvanishing curvature. The *evolute* of α is the curve
$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t),$$
where N is the Frenet normal.

   (a) Show that the tangent line to $\beta(t)$ is exactly the normal line to $\alpha(t)$.

   (b) Let $\mathrm{nl}_0(u) = \alpha(0) + uN(0)$ be the normal line to $\alpha(0)$ and let $\mathrm{nl}_t(u) = \alpha(t) + uN(t)$ be the normal line to $\alpha(t)$. Let $I(0, t)$ be the point where nl_0 and nl_t intersect. Prove that
$$\lim_{t \to 0} I(0, t) = \beta(0).$$

6. (Shifrin Problem 1.2.11) Suppose $\alpha(t)$ is a regular curve, not necessarily parametrized by arclength. Show that the torsion of α is given by
$$\tau = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{\|\alpha' \times \alpha''\|^2}.$$

7. (Shifrin Problem 1.2.20) Two distinct parametrized curves α and β are called *Bertrand mates* if, for each t, the normal line to α at $\alpha(t)$ is the same as the normal line to β at $\beta(t)$.

   Suppose α and β are Bertrand mates.

   (a) If α is parametrized by arclength, show that $\beta(s) = \alpha(s) + rN(s)$ for some constant r, meaning that corresponding points on α and β are a constant distance apart.

   (b) Show that, moreover, the angle between the tangent vectors to α and β at corresponding points is constant. (Hint: consider the dot product)

(c) Suppose α is parametrized by arclength and that both κ and τ are nonvanishing. Show that α has a Bertrand mate β if and only if there are constants r and c so that $r\kappa + c\tau = 1$.

(d) Show that a curve α that has more than one Bertrand mate must be a helix (and hence have infinitely many Bertrand mates).

8. Recall that the Bishop frame for a curve $\alpha(s)$ defines associated quantities $k_1(s)$ and $k_2(s)$. The parametrized plane curve $(k_1(s), k_2(s))$ is called the *normal development* of the curve α. Compute the Bishop frame and normal development of a helix.

   (Hint: One strategy is to use your answer to HW 1, Problem 2 to compute the Frenet frame of the helix, then write the first Bishop normal vector N_1 as

   $$N_1(s) = \cos(\theta(s))N(s) + \sin(\theta(s))B(s),$$

   where $\theta(s)$ is unknown. Use the fact that $N_1'(s)$ is parallel to $T(s)$ to find an equation for $\theta'(s)$, integrate to find $\theta(s)$, and plug the result into the above equation.)