Math 474 HW #1

Due 2:00 PM Friday, Sept. 5

Choose 3 of the following 5 problems to write up and turn in.

1. A circular disk of radius 1 in the xy-plane rolls without slipping along the x-axis. A point on the rim of the disk traces out a curve called a *cycloid.*
   (a) Find a parametrization $\alpha(t)$ for the cycloid.
   (b) What is the length of the portion of the cycloid corresponding to one complete revolution of the disk?

2. (Shifrin Problem 1.1.2) Consider the helix $\alpha(t) = (a\cos t, a\sin t, bt)$ where a and b are positive real numbers. Calculate $\alpha'(t)$ and $\|\alpha'(t)\|$, and then reparametrize α by arclength.

3. Given a regular curve $\alpha(s)$ parametrized by arclength, the curve $t(s)$ of unit tangent vectors is a curve on the unit sphere called the *tangent indicatrix.* Likewise, the curve $n(s)$ of unit normals is a curve on the unit sphere called the *normal indicatrix* and the curve $b(s)$ of binormals is called the *binormal indicatrix.*
   (a) Prove that the speed of the tangent indicatrix is equal to the curvature of α.
   (b) Prove that the speed of the normal indicatrix is equal to the length of the vector $\begin{pmatrix}\kappa(s)\\ \tau(s)\end{pmatrix} \in \mathbb{R}^2$, where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of α.
   (c) Prove that the speed of the binormal indicatrix is equal to $|\tau(s)|$.

4. (similar to Shifrin Problem 1.1.8) Let $p, q \in \mathbb{R}^3$ and let $\alpha : [a, b] \to \mathbb{R}^3$ be a parametrized curve such that $\alpha(a) = p$ and $\alpha(b) = q$.
   (a) Show that, for any unit vector v,
   $$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v \, \mathrm{dt} \leq \int_a^b \|\alpha'(t)\| \, \mathrm{dt}\,.$$
   (b) Let $v = \frac{q-p}{\|q-p\|}$ and use part (a) to prove that
   $$\|q - p\| \leq \text{length}(\alpha).$$
   In other words, the shortest path from p to q is the straight line!

5. Assume $\alpha(s)$ is an arclength parametrized regular curve with associated Frenet frame (t, n, b). Find a vector $u(s)$ so that
   $$\begin{aligned} t'(s) &= u(s) \times t(s) \\ n'(s) &= u(s) \times n(s) \\ b'(s) &= u(s) \times b(s). \end{aligned}$$
   The vector $u(s)$ is called the *Darboux vector* for α. Find an expression for the length of the Darboux vector in terms of the curvature and torsion of α.