Math 469 HW #5

Due 1:00 PM Friday, Feb. 28

1. (Axler Problem 3.E.15) Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim(V/(\operatorname{null}\varphi)) = 1$.

2. (Axler Problem 3.E.16) Suppose U is a subspace of V so that $\dim(V/U) = 1$. Prove that there exists $\varphi \in \mathcal{L}(V, \mathbb{F})$ so that $\operatorname{null}\varphi = U$.

3. (Axler Problem 3.F.3) Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ so that $\varphi(v) = 1$.

Bonus Problem (Axler Problem 3.F.34) The *double dual space* of V, denoted V'', is defined to be the dual space of V'; i.e., $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

1. Show that $\Lambda \in \mathcal{L}(V, V'')$.
2. Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.
3. Show that if V is finite-dimensional, then Λ is an isomorphism from V to V''.

   [*When V is finite-dimensional, we know V is isomorphic to V', but actually writing down such an isomorphism requires choosing a basis. In contrast, the isomorphism Λ from V to V'' does not require the choice of a basis, and hence is considered more natural. Conceptually, this is like the physics preference for theories that can be expressed in a coordinate-free way.*]