Math 419 HW #7

Due 3:00 PM Friday, Oct. 25

1. (Jones Problem 4–1 (2.)) Suppose f has a pole at z_0 of order $\leq N$. The function $(z-z_0)^N f(z)$ has a removable singularity at z_0. Prove that

$$\operatorname{Res}(f, z_0) = \left. \frac{\left(\frac{d}{dz}\right)^{N-1} \left[(z-z_0)^N f(z)\right]}{(N-1)!} \right|_{z=z_0}$$

2. Show that, for $n \geq 1$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.$$

(*Note:* The right hand side turns out to be $\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\sqrt{\pi}$, where Γ is the gamma function.[1] As $n \to \infty$, this quantity is asymptotic to $\sqrt{\pi n}$.)

[1] Relating to what we've been talking about in class, the gamma function can be defined as the Mellin transform of e^{-x}: for $z \in \mathbb{C}$, $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x$.