Math 419 HW #2

Due 3:00 PM Friday, Sept. 6

1. Let $\langle \cdot, \cdot \rangle$ be the usual inner product (dot product) on $\mathbb{R}^2$, meaning that if $Z = (x, y)$ and $W = (u, v)$, then
$$\langle Z, W \rangle = xu + yv.$$
On the other hand, we can define a *Hermitian inner product* on $\mathbb{C}$ by
$$\langle\!\langle z, w \rangle\!\rangle = z\overline{w}.$$
The word *Hermitian* is used to indicate that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is not symmetric, but rather satisfies the equation
$$\langle\!\langle z, w \rangle\!\rangle = \overline{\langle\!\langle w, z \rangle\!\rangle}.$$
Show that
$$\langle z, w \rangle = \frac{1}{2}\left(\langle\!\langle z, w \rangle\!\rangle + \langle\!\langle w, z \rangle\!\rangle\right) = \mathrm{Re}(\langle\!\langle z, w \rangle\!\rangle),$$
where, as usual, we're identifying a complex number $z = x + iy$ with the vector $(x, y) \in \mathbb{R}^2$.

2. (Jones Problem 1–8) Let C be the circle in $\mathbb{C}$ with center $a \in \mathbb{C}$ and radius $r > 0$. From p. 8 in Jones we know that a complex number z lies on C if and only if
$$|z|^2 - 2\,\mathrm{Re}(z\overline{a}) + |a|^2 = r^2. \qquad (\star)$$
The goal is to understand what happens to C when we apply the function $f(z) = \frac{1}{z}$.

   (a) If $0 \notin C$, define
$$D = \left\{ \frac{1}{z} : z \in C \right\}.$$
   Prove that D is also a circle, and calculate its center and radius.

   (b) If $0 \in C$, then define
$$D = \left\{ \frac{1}{z} : z \in C, z \neq 0 \right\}.$$
   Describe D geometrically and prove your description is correct.

   (*Hint for both parts:* If a complex number w is in D, then we know $w = \frac{1}{z}$ for some $z \in C$, meaning that $z = \frac{1}{w} \in C$. Use $(\star)$ to see what this tells you about w and think about what it would mean to show that w lies on a circle. It will be useful to come up with a characterization of $0 \in C$ in terms of a and r.)