Math 369: Day 9

Application: Electrical Circuits

One place linear algebra turns out to be very useful is in analysis of networks. One example is in electrical circuits, where the key laws are:

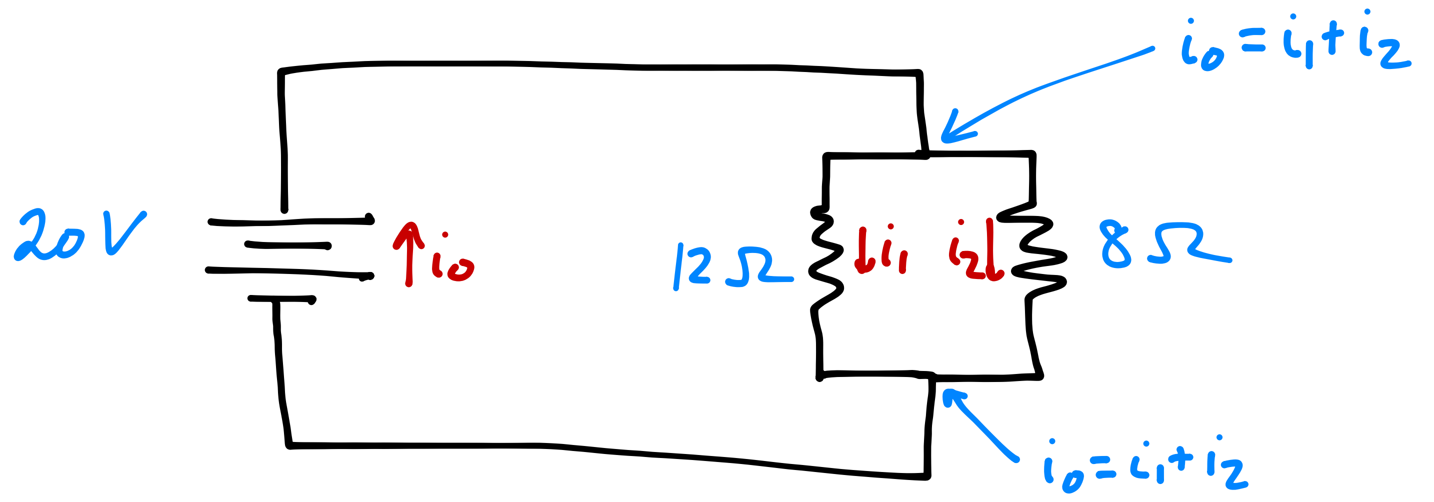
Ohm's Law: The voltage drop V due to a current I traversing a resistor w/ resistance R is

$$V = IR$$

Kirchhoff's Current Law: For any point in the network, the flow in equals the flow out.

Kirchhoff's Voltage Law: Around any closed loop, the net voltage drop is 0.

Ex:



"Inner" loop: $20 - 12 i_1 = 0$

"Outer" loop: $20 - 8 i_2 = 0$

Small loop: $12 i_1 = 8 i_2$ or $12 i_1 - 8 i_2 = 0$

So to determine i_0, i_1, i_2, need to solve the (redundant) system

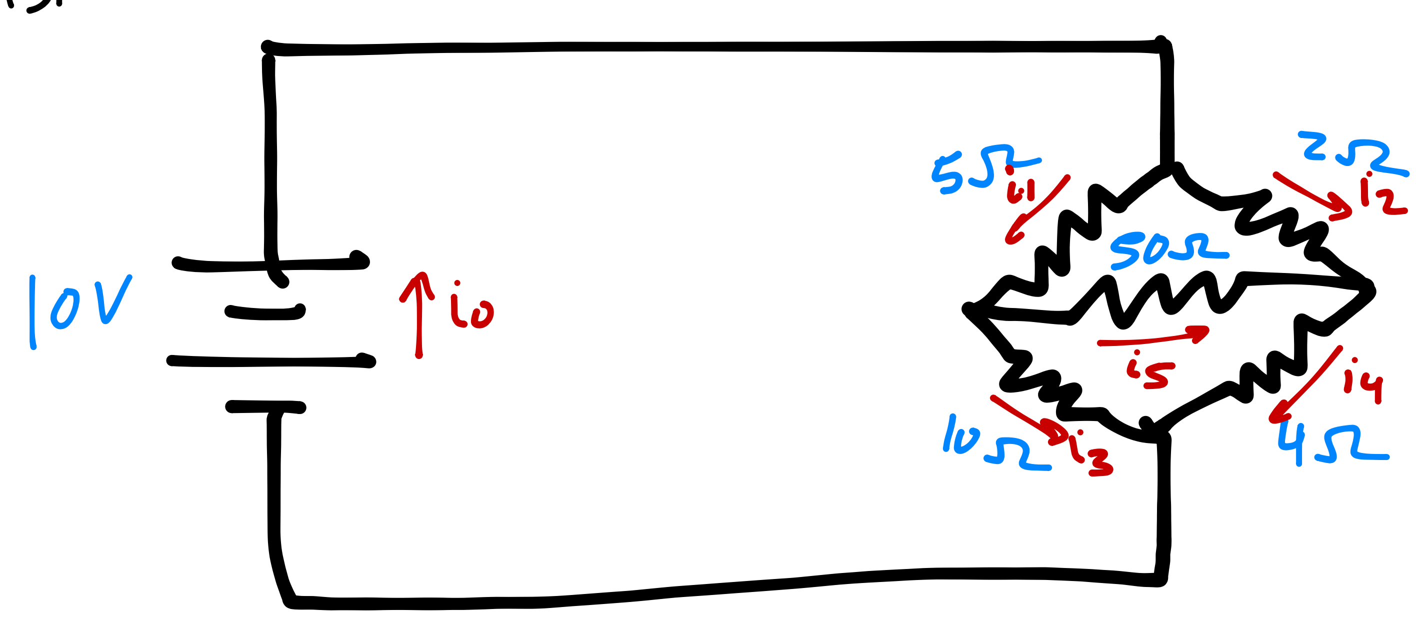
$$\begin{aligned} i_0 - i_1 - i_2 &= 0 \\ i_0 - i_1 - i_2 &= 0 \\ 12 i_1 \quad &= 20 \\ 8 i_2 &= 20 \\ 12 i_1 - 8 i_2 &= 0 \end{aligned}$$

Obviously, we don't actually need any linear algebra to solve this: $i_1 = \frac{20}{12} = \frac{5}{3}$

$i_2 = \frac{20}{8} = \frac{5}{2}$

$i_0 = i_1 + i_2 = \frac{25}{6}$

Ex: Wheatstone bridge



$i_0 = i_1 + i_2$

$i_0 = i_3 + i_4$

$i_1 = i_3 + i_5$

$i_2 + i_5 = i_4$

$5 i_1 + 10 i_3 = 10$

$2 i_1 + 4 i_4 = 10$

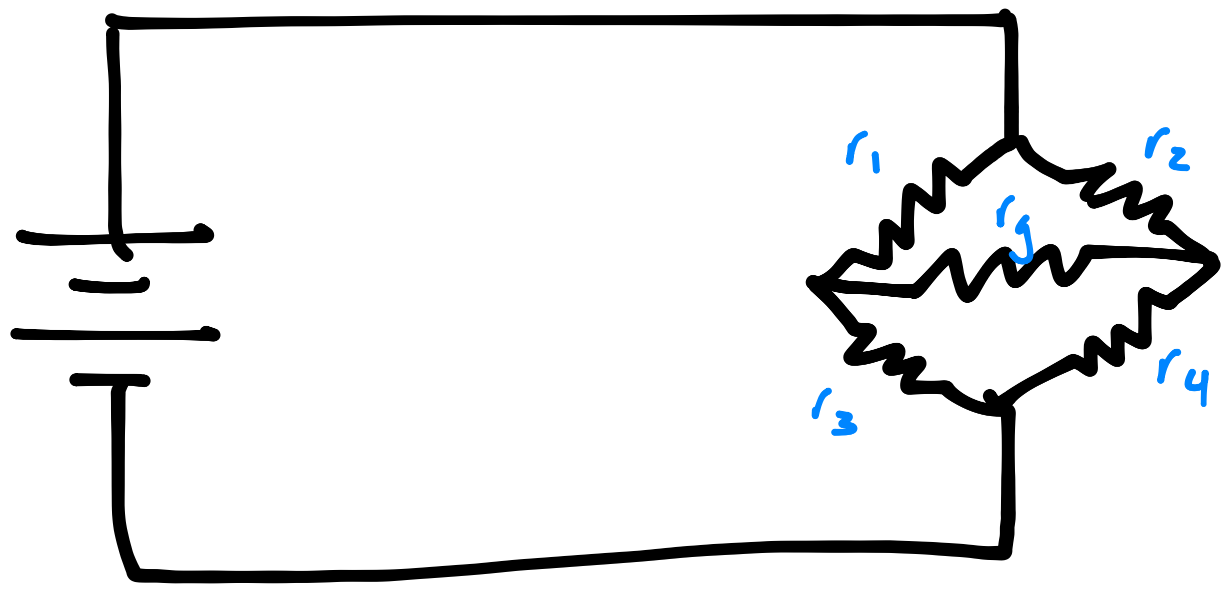
$5 i_1 + 50 i_5 - 2 i_2 = 0$

$50 i_5 + 4 i_4 - 10 i_3 = 0$

$$\left[\begin{array}{cccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & 0 & 10 & 0 & 0 & 10 \\ 0 & 2 & 0 & 0 & 4 & 0 & 10 \\ 0 & 5 & -2 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & -10 & 4 & 50 & 0 \end{array}\right]$$

Solution (after some work): $i_0 = 7/3,\ i_1 = 2/3,\ i_2 = 5/3,\ i_3 = 2/3,\ i_4 = 5/3,\ i_5 = 0$

Prop: For a general Wheatstone bridge:



If the current through r_g is 0 then $\frac{r_2}{r_1} = \frac{r_4}{r_3}$

(In practice, one doesn't know r_4 & puts an ammeter at r_g, then adjusts $r_1, r_2,$ & r_3 until the current at the meter is 0, then calculates $r_4 = \frac{r_2 r_3}{r_1}$)

Pf: Using Kirchhoff's voltage law, $i_1 r_1 + i_g r_g - i_2 r_2 = 0$ & $i_4 r_4 + i_g r_g - i_3 r_3 = 0$

Clearly, if $i_g = 0$ we get $i_2 r_2 = i_1 r_1 \iff \frac{r_2}{r_1} = \frac{i_1}{i_2}$ & also $i_3 r_3 = i_4 r_4 \iff \frac{r_4}{r_3} = \frac{i_3}{i_4}$.

But now $i_3 = i_1$ & $i_4 = i_2$, so $\frac{r_2}{r_1} = \frac{i_1}{i_2} = \frac{i_3}{i_4} = \frac{r_4}{r_3}$.

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