Math 309: Day 8

We saw last time that if A is an $n\times n$ invertible matrix, then the range of $T_A:\mathbb{R}^n\to\mathbb{R}^n$ is all of $\mathbb{R}^n$, since

$A\vec{x}=\vec{b}$ is solvable for any $\vec{b}\in\mathbb{R}^n$ (specifically, $\vec{x}=A^{-1}\vec{b}$ is the solution).

Moreover, $T_{A^{-1}}$ is also a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, & notice that $T_{A^{-1}}(T_A(\vec{x}))=A^{-1}Ax=x$ & $T_A(T_{A^{-1}}(\vec{x}))=AA^{-1}\vec{x}=\vec{x}$, so

$T_{A^{-1}}=(T_A)^{-1}$, the inverse mapping of T_A.

If A is $m\times n$ w/ $m\neq n$, then $T_A:\mathbb{R}^n\to\mathbb{R}^m$. Since A not square, A can't be invertible, but there is still a way to get a mapping that goes "backwards" from $\mathbb{R}^m\to\mathbb{R}^n$: take the transpose A^T which interchanges rows & columns of A.

Ex: $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$, so $T_A:\mathbb{R}^3\to\mathbb{R}^2$. But then $A^T=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$ & $T_{A^T}:\mathbb{R}^2\to\mathbb{R}^3$.

Notice: $T_{A^T}(T_A(\vec{x}))=A^TA\vec{x}=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}17&22&27\\22&29&36\\27&36&45\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$T_A(T_{A^T}(\vec{y}))=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}14&32\\32&77\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$

symmetric

Df: A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a map $T:\mathbb{R}^n\to\mathbb{R}^m$ so that

1. T is additive: For all $\vec{u},\vec{v}\in\mathbb{R}^n$, $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$.
2. T is homogeneous: For all $\vec{u}\in\mathbb{R}^n$ & $\lambda\in\mathbb{R}$, $T(\lambda u)=\lambda Tu$.

Ex: $T:\mathbb{R}^2\to\mathbb{R}^3$ given by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x+3y\\x-y\\x+y\end{bmatrix}$

Ex: $T_A:\mathbb{R}^n\to\mathbb{R}^m$, where A is $m\times n$.

Ex: $(x,y)\mapsto(xy,\,x^2-y^2)$ is not.

Now, I claim that if $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation, then there exists an $m\times n$ matrix A so that $T=T_A$.

Why? Suppose $\vec{v}=\begin{bmatrix}v_1\\\vdots\\v_n\end{bmatrix}\in\mathbb{R}^n$. Then $\vec{v}=v_1\underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{\vec{e}_1}+v_2\underbrace{\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}}_{\vec{e}_2}+\dots+v_n\underbrace{\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}}_{\vec{e}_n}$, so

$$T(\vec{v})=T(v_1\vec{e}_1+\dots+v_n\vec{e}_n)=T(v_1\vec{e}_1)+\dots+T(v_n\vec{e}_n)=v_1T(\vec{e}_1)+\dots+v_nT(\vec{e}_n)$$

But now remember that when we have a matrix A, then $A\vec{v}$ is the linear combination of the cols of A w/ coeffs given by the entries in $\vec{v}$.

So just define A so that the columns of A are $T(\vec{e}_1),T(\vec{e}_2),\dots,T(\vec{e}_n)$.

Ex: Think of the example $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+y \end{bmatrix}$

Then the preceding reasoning says we should look at $T(\vec{e}_1)$ & $T(\vec{e}_2)$ where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(1)+3(0) \\ 1-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2(0)+3(1) \\ 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

So the claim is that $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

So let's check that $T = T_A$: $T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ x-y \\ x+y \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ ✓

So this not only shows that all linear transformations come from matrices, but gives us a procedure for determining the matrix just from knowledge of the transformation (& really just from knowledge of what the transformation does to a small # of vectors)