Math 369: Day 7

So far we've just been thinking of matrices as ways of encoding systems of equations, but another way to think of them is as transformations.

Suppose A is an $m\times n$ matrix. Then we know that we can multiply A by an n-dimensional vector $\vec{v}\in\mathbb{R}^n$ & get a new vector $A\vec{v}$ which is m-dimensional.

In other words, A induces a mapping $T_A:\mathbb{R}^n\to\mathbb{R}^m$ where $T_A(\vec{v})=A\vec{v}$; it's a machine for turning n-dim'l vectors into m-dim'l vectors.

This mapping has nice properties: ① $T_A(\vec{0})=\vec{0}$ b/c $A\vec{0}=\vec{0}$

② $T_A(c\vec{v})=cT_A(\vec{v})$ b/c $A(c\vec{v})=cA\vec{v}$

③ $T_A(\vec{v}+\vec{w})=T_A(\vec{v})+T_A(\vec{w})$ b/c $A(\vec{v}+\vec{w})=A\vec{v}+A\vec{w}$.

Together, these mean that T_A is a linear transformation.

What does T_A do to a vector $\vec{v}$?

As an algebraic expression, this is kind of obvious: you get whatever formula you get for $A\vec{v}$.

Ex: $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$. Then if $\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$, $T_A(\vec{v})=A\vec{v}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}v_1+2v_2\\3v_1+4v_2\end{bmatrix}$

What does this mean? Well, notice that $\begin{bmatrix}v_1+2v_2\\3v_1+4v_2\end{bmatrix}=v_1\begin{bmatrix}1\\3\end{bmatrix}+v_2\begin{bmatrix}2\\4\end{bmatrix}$, so T_A sends $\vec{v}$ to the linear combination of the columns of A with coefficients given by the entries of $\vec{v}$.

So another way to look at an equation like $A\vec{x}=\vec{b}$ is to say that this says $\vec{b}$ can be written as a linear combination of the columns of A.

So for example if we solve $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\11\end{bmatrix}$, we see that $x_1=1$, $x_2=2$, & so $\begin{bmatrix}5\\11\end{bmatrix}=1\begin{bmatrix}1\\3\end{bmatrix}+2\begin{bmatrix}2\\4\end{bmatrix}$.

So now we might ask questions like: is there some linear combination of the vectors $\begin{bmatrix}1\\3\\5\end{bmatrix}$ & $\begin{bmatrix}2\\4\\6\end{bmatrix}$ that produces the vector $\begin{bmatrix}-1\\1\\3\end{bmatrix}$?

Well, if there were, we'd have $\begin{bmatrix}-1\\1\\3\end{bmatrix}=x\begin{bmatrix}1\\3\\5\end{bmatrix}+y\begin{bmatrix}2\\4\\6\end{bmatrix}$ for some x & y, so

$$\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}x+2y\\3x+4y\\5x+6y\end{bmatrix} = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},$$

so it boils down to solving $\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1\\1\\3\end{bmatrix}$.

If $A = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$ were invertible, we'd know there is a solution, namely $\vec{x} = A^{-1}\begin{bmatrix}-1\\1\\3\end{bmatrix}$. But A isn't a square matrix so it can't be invertible, & need to solve another way:

$$\sim \left[\begin{array}{cc|c}1&2&-1\\3&4&1\\5&6&3\end{array}\right] \xrightarrow[{[3]\to[3]+(-5)[1]}]{{[2]\to[2]+(-3)[1]}} \left[\begin{array}{cc|c}1&2&-1\\0&-2&4\\0&-4&8\end{array}\right] \Rightarrow \left.\begin{matrix}-2x_2=4\\-4x_2=8\end{matrix}\right\} \Rightarrow x_2=-2 \Rightarrow x_1+2x_2=-1$$

so $x_1+2(-2)=-1$
$x_1-4=-1$
$x_1=3$

so $x_1=3$, $x_2=-2$ work, & indeed we see that $3\begin{bmatrix}1\\3\\5\end{bmatrix}-2\begin{bmatrix}2\\4\\6\end{bmatrix} = \begin{bmatrix}-1\\1\\3\end{bmatrix}$

In terms of the transformation T_A, a vector $\vec{b}\in\mathbb{R}^m$ can either be in the range of T_A or not, corresponding to whether $A\vec{x}=\vec{b}$ is solvable or not or, equivalently, to whether $\vec{b}$ can be written as a linear combination of the columns of A or not.

If A is invertible, then $T_A:\mathbb{R}^n\to\mathbb{R}^n$ & $A\vec{x}=\vec{b}$ is always solvable, so the range of T_A is all of $\mathbb{R}^n$.