Math 369: Day 6

Last time we saw that, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Ex: What is the solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the eqn $\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}_{\vec{b}}$?

A: $A^{-1} = \frac{1}{2\cdot2-3\cdot1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & \cdot3 \\ -1 & 2 \end{bmatrix}$, so $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Check: $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ ✓

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. What are solns to $A\vec{x} = \vec{b}$?

$A^{-1} = \frac{1}{1\cdot4-2\cdot2}\begin{bmatrix} 4 & -2 \\ \cdot2 & 1 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ DNE!

So A is not invertible... it is a singular matrix. ... Row reduce $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array}\right] \xrightarrow{[2]\to[2]+(-2)[1]} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 + 2x_2 = 3 \Rightarrow x_1 = 3-2x_2$, so soln's are $(x_1, x_2) = (3-2x_2, x_2)$.

Now, to get $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, we had to solve the 4×4 system $\left[\begin{array}{cccc|c} a & c & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{array}\right]$, which is worse than just solving $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ... so what's the point?

Well, for one thing, suppose we wanted to solve $A\vec{x} = \vec{b}$, $A\vec{x} = \vec{c}$, $A\vec{x} = \vec{d}$, $A\vec{x} = \vec{f}$, etc.

Pretty quickly it becomes easier to just solve for A^{-1} once & for all & then get solutions $\vec{x} = A^{-1}\vec{b}$, $\vec{x} = A^{-1}\vec{c}$, etc.

General Procedure for determining A^{-1} (when it exists)

By example.

Ex: $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ write the super-augmented matrix $[A \mid I] = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ & put in RREF form

$\xrightarrow[{[3]\to[3]+[1]}]{[2]\to[2]+(-1)[1]} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap } [2] \,\&\, [3]} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{[2]\to-[2]} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$

$\xrightarrow{[1]\to[1]+[2]} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[{[2]\to[2]+2[3]}]{[1]\to[1]+[3]} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$ then the result is $[I \mid A^{-1}]$, so $A^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -3 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

Ex: Solve $A\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Then $\vec{x} = A^{-1}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -3 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ -3 \end{bmatrix}$

Ex: Solve $A\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Then $\vec{x} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -3 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a+b-c \\ -3a+2b-c \\ -a+b \end{bmatrix}$

So no matter what a, b, and c are, we are guaranteed a solution!

This is generally true: if A is invertible, then $A\vec{x} = \vec{b}$ can always be solved, no matter what $\vec{b}$ is: there's exactly one sol'n & it is $\vec{x} = A^{-1}\vec{b}$.