Math 369: Day 5

As we saw last time, if we can figure out what the inverse A^{-1} of a matrix A is, we'll be able to solve $A\vec{x}=\vec{b}$

by multiplying both sides by A^{-1}:
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

So what's A^{-1}? It's supposed to be the matrix that undoes mult. by A; in other words, $A^{-1}A$ should be a matrix that does

nothing when you multiply it by some vector $\vec{x}$.

So that means we're looking for a vector I so that $I\vec{x}=\vec{x}$. Of course, if $\vec{x}$ is n-dim'l, that means I had

better be $n\times n$ ... a square matrix. Now, if $I=(d_{ij}) = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}$, then we know

$$I\vec{x} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_{11}x_1 + \cdots + d_{1n}x_n \\ \vdots \\ d_{n1}x_1 + \cdots + d_{nn}x_n \end{bmatrix}$$

but this is supposed to equal $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ ... the only way to make this work is if $I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$, which is

called the $n\times n$ identity matrix.

So now we're looking for A^{-1} so that $A^{-1}A = I$.

First, we need to talk about multiplying matrices, which is actually easy. We already know how to mult. a matrix by a vector...

but now if I want to mult. a matrix A by a matrix B, I can just think of the columns of B as vectors, mult.

each individually by A, & then line the results up.

If A is $m\times n$, we know we can only multiply A by n-dim'l vectors, so the columns of B had better be n-dim'l, so

B must have n rows, meaning B is an $n\times p$ matrix for some p.

Ex: $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 9 \\ 20 & 19 & 21 \end{bmatrix}$

since $A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}$, $A\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \end{bmatrix}$ & $A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$.

In general, if $A=(a_{ij})$, $B=(b_{ij})$ where A is $m\times n$ & B is $n\times p$, then $C=AB$ is an $m\times p$ matrix w/ entries

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$$

This only makes sense if the # of cols of A equals the number of rows of B.

So now we know what $A^{-1}A=I$ should mean. It turns out that, just as $\frac{1}{a}$ doesn't always make sense

(what if $a=0$?), A^{-1} also doesn't always make sense. Let's do some counting to see when it's even plausible.

If A is $m\times n$, we know A^{-1} must be $p\times m$ for some p for $A^{-1}A$ to even make sense, so $A^{-1}A$ is $p\times n$.

But we know I is a square matrix, so in fact we must have $p=n$ so that $A^{-1}A=I$ can be $n\times n$ $\Rightarrow A^{-1}$ is $n\times m$.

Also, if $A=(a_{ij})$ & $A^{-1}=(y_{kl})$, then $A^{-1}A=I$ becomes $\begin{bmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nm}\end{bmatrix}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1\end{bmatrix}$

or $$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1\end{bmatrix}}_{n\times n}=\begin{bmatrix} \sum_{i=1}^{m} y_{1i}a_{i1} & \sum_{i=1}^{m} y_{1i}a_{i2} & \cdots & \sum_{i=1}^{m} y_{1i}a_{in} \\ \vdots & & & \vdots \\ \sum_{i=1}^{m} y_{ni}a_{i1} & \sum_{i=1}^{m} y_{ni}a_{i2} & \cdots & \sum_{i=1}^{m} y_{ni}a_{in}\end{bmatrix}$$

So this is $n\cdot n=n^2$ equations in $n\cdot m=nm$ variables y_{kl}. We should only really expect a unique solution when

the number of variables equals the number of eqns, so when $n=m$.

So the first major restriction is that only square matrices can have inverses!

In other words, if $A\vec{x}=\vec{b}$ w/ A being $n\times n$ then we could try to find A^{-1} & get the solution $\vec{x}=A^{-1}\vec{b}$, but if

A is $m\times n$ with $m\neq n$, we can forget about this approach.

So now let's do this in the case A is 2×2.

So suppose $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ & we want to determine $A^{-1}=\begin{bmatrix} x & y \\ z & w\end{bmatrix}$. We know $A^{-1}A=I$, so we have

$$\begin{bmatrix} x & y \\ z & w\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

or
$$\begin{bmatrix} ax+cy & bx+dy \\ az+cw & bz+dw\end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},$$

So x, y, z, w are solutions to the system

$$\begin{aligned} ax + cy &= 1 \\ bx + dy &= 0 \\ az + cw &= 0 \\ bz + dw &= 1 \end{aligned}$$

which corresponds to the augmented matrix

$$\left[\begin{array}{cccc|c} a & c & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{array}\right]$$

$$\xrightarrow{[1] \to \frac{1}{a}[1]} \left[\begin{array}{cccc|c} 1 & c/a & 0 & 0 & 1/a \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{array}\right] \xrightarrow{[2] \to [2] + (-b)[1]} \left[\begin{array}{cccc|c} 1 & c/a & 0 & 0 & 1/a \\ 0 & d - \frac{bc}{a} & 0 & 0 & -b/a \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{array}\right] \xrightarrow{[3] \to \frac{1}{a}[3]} \left[\begin{array}{cccc|c} 1 & c/a & 0 & 0 & 1/a \\ 0 & d - \frac{bc}{a} & 0 & 0 & -b/a \\ 0 & 0 & 1 & \frac{c}{a} & 0 \\ 0 & 0 & b & d & 1 \end{array}\right]$$

$$\xrightarrow{[4] \to [4] + (-b)[3]} \left[\begin{array}{cccc|c} 1 & c/a & 0 & 0 & 1/a \\ 0 & d - \frac{bc}{a} & 0 & 0 & -b/a \\ 0 & 0 & 1 & \frac{c}{a} & 0 \\ 0 & 0 & 0 & d - \frac{bc}{a} & 1 \end{array}\right]$$

so $\left(d - \frac{bc}{a}\right) w = 1 \Rightarrow w = \dfrac{1}{d - \frac{bc}{a}} = \dfrac{1}{\frac{ad - bc}{a}} = \dfrac{a}{ad - bc}$

& $\left(d - \frac{bc}{a}\right) y = -\frac{b}{a} \Rightarrow y = \dfrac{-b/a}{d - \frac{bc}{a}} = \dfrac{-b/a}{\frac{ad-bc}{a}} = -b$

Then $0 = z + \frac{c}{a} w = z + \frac{c}{a}\left(\frac{a}{ad-bc}\right) = z + \frac{c}{ad-bc} \Rightarrow z = \frac{-c}{ad-bc}$

& $\frac{1}{a} = x + \frac{c}{a} y = x + \frac{c}{a}\left(\frac{-b}{ad-bc}\right) = x - \frac{bc}{a(ad-bc)} \Rightarrow x = \frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{ad-bc}{a(ad-bc)} + \frac{bc}{a(ad-bc)}$

$$= \frac{ad}{a(ad-bc)} = \frac{d}{ad-bc}$$

So we see that $A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Check</u>: $A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓