Math 369: Day 3

We left off last time in the middle of the following example:

Ex:

$$\begin{aligned} x_3 - x_4 - x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 3x_4 + 3x_5 &= 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6 \end{aligned}$$

We had translated into a matrix & reduced to

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 4 \end{array}\right] \xrightarrow{\text{Swap } [2] \,\&\, [4]} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & -2 & 0 \end{array}\right] \xrightarrow[{[4] \to [4] + 2[2]}]{[3] \to [3] + 2[2]} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & -1 & -3 & 8 \\ 0 & 0 & 0 & 0 & -4 & 8 \end{array}\right]$$

$\Rightarrow$ $-4x_5 = 8 \Rightarrow \boxed{x_5 = -2}$

so $-x_4 - 3(-2) = 8 \Rightarrow -x_4 = 2 \Rightarrow \boxed{x_4 = -2}$

so $x_3 - (-2) - (-2) = 4 \Rightarrow x_3 + 4 = 4 \Rightarrow \boxed{x_3 = 0}$

so $x_1 + 2x_2 + 2(0) + (-2) + 2(-2) = 2 \Rightarrow x_1 + 2x_2 - 6 = 2$

$\Rightarrow x_1 + 2x_2 = 8$

$\Rightarrow \boxed{x_1 = -2x_2 + 8}$

and x_2 is a free variable.

Def: A matrix is in row echelon form if

- Any zero row lies below every nonzero row
- The first nonzero entry in each row is strictly to the right of the first nonzero entry in the row above

(These 2 imply that the first nonzero entry in each row has only zeros below.)

A column containing the first nonzero entry of a row is called a pivot column.

Now, it's a general fact that you can put any matrix into row echelon form using our 3 moves.

Moreover, putting the augmented matrix of a system in row echelon form makes solving the system really easy.

Ex:

$$\begin{aligned} x + 2y + 3z &= 4 \\ -x - y + 2z &= 2 \\ x + 3y + 8z &= 6 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ -1 & -1 & 2 & 2 \\ 1 & 3 & 8 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & -4 \end{array}\right) \Rightarrow 0x + 0y + 0z = -4 \quad (\Rightarrow \Leftarrow)$$

So no sol'n.

In fact, systems of linear equations are really just special cases of matrix equations, which we now describe.

First off, we will use the (standard) notation where $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers.

Def: Given a natural number n, $\mathbb{R}^n$ is the set of n-dimensional vectors w/ entries in $\mathbb{R}$:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : \text{each } x_i \in \mathbb{R} \right\}$$

Two vectors are equal if all their entries are equal.

Now, just as with numbers, we can add two vectors:

Ex: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 2+5 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$

In general, $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$. However, if $n \neq m$, then $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ makes no sense!

We can't quite multiply vectors, but we can scale them:

If $\lambda \in \mathbb{R}$ & $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, then $\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$.

So $\pi \begin{pmatrix} -1 \\ \sqrt{2} \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -\pi \\ \sqrt{2}\pi \\ 3\pi \\ -4\pi \end{pmatrix}$.