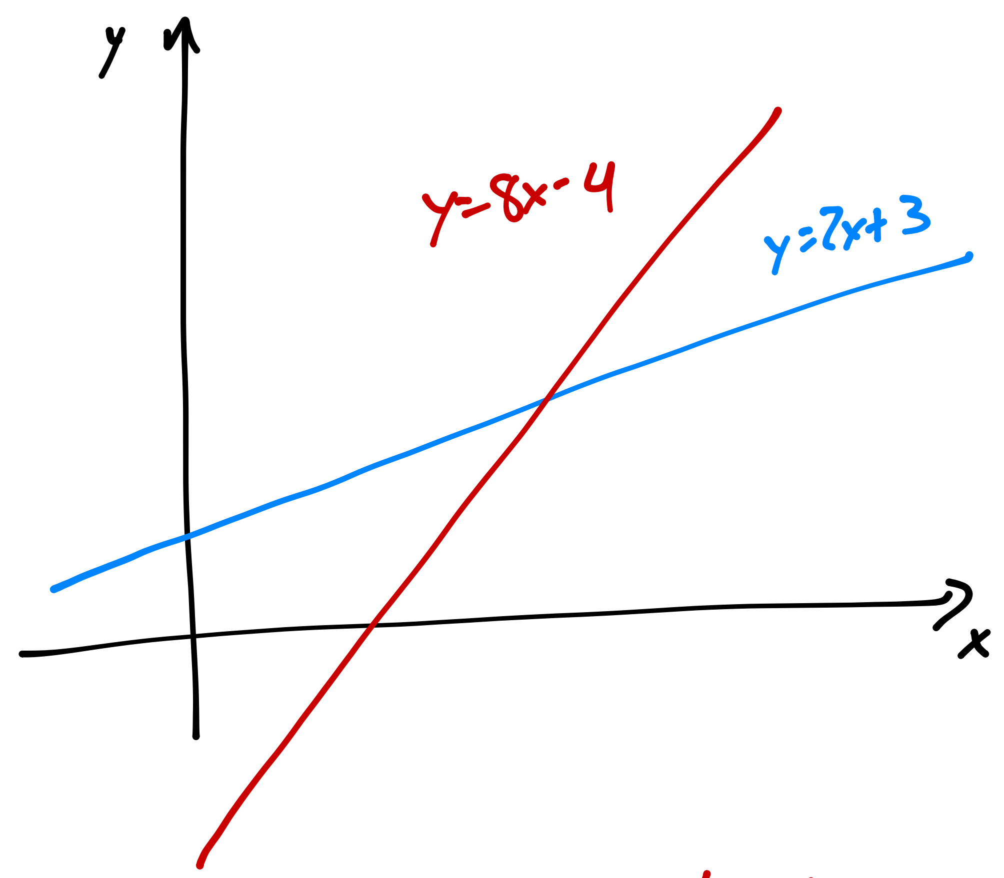
Math 369: Day 2

Solving systems of equations

Recall our system from last time $y=2x+3$

$y=8x-4$



In general, though, we might have 100 (or more) equations in 100 variables, so we need an algorithm that we can teach a computer to do.

$$\begin{aligned}2x-y&=-3\\8x-y&=4\end{aligned} \xrightarrow[\text{to eqn 2}]{\text{Add }(-4)\text{ times eqn 1}} \begin{aligned}2x-y&=-3\\3y&=16\end{aligned}$$

so $\boxed{y=\frac{16}{3}}$ and, substituting in eqn 1, $2x-\left(\frac{16}{3}\right)=-3$

$2x=\frac{7}{3}$

$\boxed{x=7/6}$

Re-solve above system in generic form:

Thm: Given a system $\begin{aligned}f_1(x_1,\dots,x_n)&=c_1\\&\vdots\\f_m(x_1,\dots,x_n)&=c_m\end{aligned}$ of m equations in n variables, the system we get by doing any of the following are equiv. to the original system:

① Exchange 2 equations

② Multiply the j^{th} eq. by some nonzero number λ.

③ Given i,j,λ w/ $i\neq j$ & $\lambda\neq 0$, replace $f_j(x_1,\dots,x_n)=c_j$ with the equation $f_j(x_1,\dots,x_n)+\lambda f_i(x_1,\dots,x_n)=c_j+\lambda c_i$

Ex:
$$\begin{aligned}3x+2y+5z&=4\\6x-y+2z&=10\\2x-2y+z&=2\end{aligned}$$

Goal: Eliminate x from 2^{nd} & 3^{rd} rows. To make life easier, first swap eqns 1 & 3:

$$\begin{aligned}2x-2y+z&=2\\6x-y+2z&=10\\3x+2y+5z&=4\end{aligned}$$

& mult. eqn 1 by $1/2$:

$$\begin{aligned}x-y+\tfrac{1}{2}z&=1\\6x-y+2z&=10\\3x+2y+5z&=4\end{aligned}$$

Now, we can eliminate x from 2^{nd} & 3^{rd} eqns:

$$\xrightarrow[{[3]\to[3]+(-3)[1]}]{{[2]\to[2]+(-6)[1]}} \begin{aligned}x-y+\tfrac{1}{2}z&=1\\5y-z&=4\\5y+\tfrac{7}{2}z&=1\end{aligned}$$

$x=2,\ y=\frac{2}{3},\ z=-\frac{2}{3}.$

Next Goal: Eliminate y from the 3^{rd} eqn.

$$\xrightarrow[{[3]\to[3]+(-1)[2]}]{} \begin{aligned}x-y+\tfrac{1}{2}z&=1\\5y-z&=4\\\tfrac{9}{2}z&=-3\end{aligned}$$

But then $\boxed{z=\frac{2}{9}(-3)=\frac{-6}{9}=\frac{-2}{3}}$

So eqn 2 becomes $5y-\left(-\frac{2}{3}\right)=4$

$5y=\frac{10}{3}$

$\boxed{y=\frac{2}{3}}$

and then eqn 1 becomes $x-\frac{2}{3}+\frac{1}{2}\left(-\frac{2}{3}\right)=1$

$x-1=1$

$\boxed{x=2}$

$\boxed{\text{Solution: } (x,y,z)=\left(2,\frac{2}{3},-\frac{2}{3}\right)}$

Okay, that's all well & good, but notice that the names of the variables weren't really important.

We will encode the information of a linear system in an **augmented matrix**:

In the example, the matrix is
$$\left(\begin{array}{ccc|c} 3 & 2 & 5 & 4 \\ 6 & -1 & 2 & 10 \\ 2 & -2 & 1 & 2 \end{array}\right)$$

In general, given a linear system
$$\begin{array}{ccccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & c_1 \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & c_m \end{array}$$

we create the augmented matrix
$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & c_m \end{array}\right)$$

Then our 3 operations on equations have the following translation into the language of matrices:

Exchange equations $\longleftrightarrow$ exchange rows

Multiply eq. by λ $\longleftrightarrow$ Multiply row by λ

Add mult. of one eq. to another $\longleftrightarrow$ add mult. of one row to another.

Two augmented matrices are called **row equivalent** if they define eq. systems of linear eqs.

Ex:
$$\begin{aligned} x_3 - x_4 - x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 3x_4 + 3x_5 &= 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6 \end{aligned}$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 2 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right) \xrightarrow{\text{Swap } [1] \,\&\, [4]} \left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 6 & 6 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \end{array}\right] \xrightarrow{[1] \to \frac{1}{3}[1]} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \end{array}\right]$$

$$\xrightarrow[{[3] \to [3] + (-2)[1]}]{[2] \to [2] + (-2)[1]} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 4 \end{array}\right] \xrightarrow{\text{Swap } [2] \,\&\, [4]} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & -2 & 0 \end{array}\right]$$

$[3] \to [3] + 2[2]$
$[4] \to [4] + 2[2]$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & -1 & -3 & 8 \\ 0 & 0 & 0 & 0 & -4 & 8 \end{array}\right] \Rightarrow$$

$-4x_5 = 8 \Rightarrow \boxed{x_5 = -2}$

so $-x_4 - 3(-2) = 8 \Rightarrow -x_4 = 2 \Rightarrow \boxed{x_4 = -2}$

so $x_3 - (-2) - (-2) = 4 \Rightarrow x_3 + 4 = 4 \Rightarrow \boxed{x_3 = 0}$

so $x_1 + 2x_2 + 2(0) + (-2) + 2(-2) = 2 \Rightarrow x_1 + 2x_2 - 6 = 2$

$\Rightarrow x_1 + 2x_2 = 8$

$\Rightarrow \boxed{x_1 = -2x_2 + 8}$

and x_2 is a free variable.