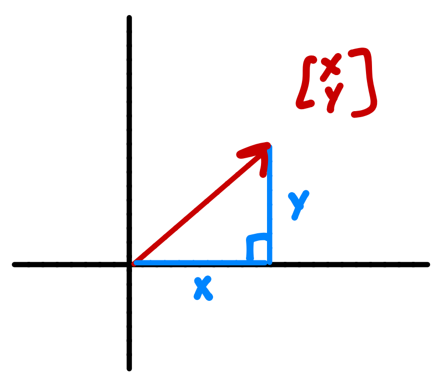
Math 369: Day 12

Norms & Dot Products

We will use $\|\vec{v}\|$ to denote the norm (or length) of a vector $\vec{v} \in \mathbb{R}^n$.

Ex: $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. What is $\|\vec{v}\|$?

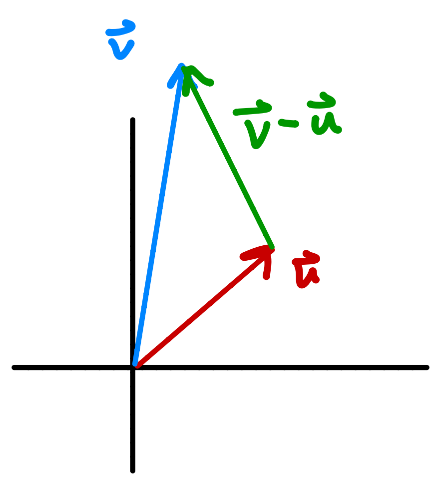


By Pythagorean Thm, $\|\vec{v}\| = \sqrt{x^2+y^2}$.

In fact, in general if $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, then $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$.

Properties:
- $\|\vec{v}\| \geq 0$ for all $\vec{v} \in \mathbb{R}^n$
- $\|\vec{v}\| = 0$ if & only if $\vec{v} = \vec{0}$.
- $\|k\vec{v}\| = |k|\,\|\vec{v}\|$ for all $k \in \mathbb{R}$ & $v \in \mathbb{R}^n$.

Now, given 2 vectors $\vec{u}$ & $\vec{v}$, what's the distance between them?



It's the length of $\vec{v} - \vec{u}$; namely $\|\vec{v} - \vec{u}\|$.

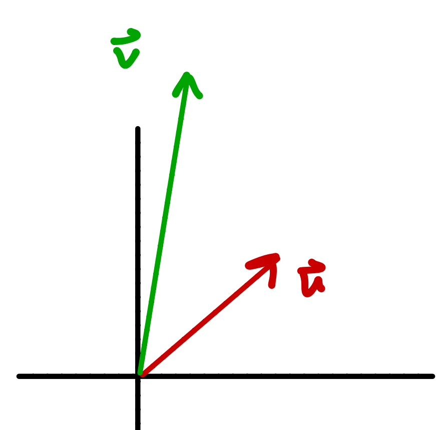
Ex: If $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$, then $\|\vec{v} - \vec{u}\| = \left\| \begin{bmatrix} -10 \\ 2 \end{bmatrix} \right\| = \sqrt{(-10)^2 + 2^2} = \sqrt{104} \approx 10.198$

Now, it's often easier to work w/ $\|\vec{v}\|^2 = v_1^2 + v_2^2 + \dots + v_n^2 = v_1v_1 + v_2v_2 + \dots + v_nv_n$.

Now we generalize a bit & define the dot product of 2 vectors to be $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$ if $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

so that $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ & $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Ex: Consider $\vec{u}, \vec{v} \in \mathbb{R}^n$ w/ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$



What does it mean if $\vec{u} \cdot \vec{v} = 0$?

$0 = u_1v_1 + u_2v_2$

$\Rightarrow v_2 = -\frac{u_1}{u_2} v_1$. Since $\vec{u}$ is on a line of slope $\frac{u_2}{u_1}$ & $\vec{v}$ is on a line of slope $-\frac{u_1}{u_2}$, we see that these 2 vectors are perpendicular.

In fact, this is generally true: if $\vec{u}, \vec{v} \in \mathbb{R}^n$ s.t. $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ & $\vec{v}$ are perpendicular.