Math 369: Day 11

Using cofactor expansion, computing the determinant of an $n\times n$ matrix takes

$$\underbrace{n}_{\substack{\text{entries in}\\ \text{a row}}}\cdot\underbrace{(n-1)}_{\substack{\text{entries in}\\ \text{each row of}\\ \text{the }(n-1)\times(n-1)\\ \text{minor}}}\cdot\underbrace{(n-2)}_{\substack{\text{entries in a row}\\ \text{of the }(n-2)\times(n-2)\\ \text{minor}}}\cdots 3\cdot 2\cdot 1 = n!$$

operations. This is obviously completely infeasible if n is very big; for example $60! \approx 8.32\times 10^{81}$, which is approximately the number of atoms in the universe. But my computer can compute the determinant of a random 60×60 matrix in less than $\frac{1}{1000}$ of a second. So how the heck does it do that?

Well, the basic idea is to take advantage of the fact that determinants are multiplicative: $\det(AB)=\det(A)\det(B)$

& to use the fact that determinants of upper triangular matrices are simple:

$$\det\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n}\\ 0 & a_{22} & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a_{33} & \cdots & a_{3n}\\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}\end{bmatrix} \underset{\text{expand on last row}}{=} 0+0+\cdots+0+(-1)^{n+n}a_{nn}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & a_{n-1,n-1}\end{vmatrix}$$

$$\underset{\text{expand on last row}}{=} a_{nn}\left(0+0+\cdots+0+a_{n-1,n-1}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & a_{n-2,n-2}\end{vmatrix}\right)$$

$$= \cdots = a_{nn}a_{n-1,n-1}\cdots a_{22}a_{11} = \text{product of diagonal entries.}$$

So then the idea is to realize the row reduction of A as a product of matrices: $\mathrm{REF}(A) = E_k E_{k-1}\cdots E_2 E_1 A$

Then $\det(\mathrm{REF}(A))$ (which is easy) equals $\det(E_k E_{k-1}\cdots E_2 E_1 A) = \det(E_k)\det(E_{k-1})\cdots\det(E_2)\det(E_1)\det(A)$.

If we know each of the $\det(E_i)$, then we can solve for $\det(A)$.

But row row operations are easy to realize as matrix multiplications: for example, on 4×4 matrices:

- Swap rows 1 & 3 is realized by $\begin{bmatrix} 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\end{bmatrix} \leftarrow \det = -1$

- Mult. row 2 by k is realized by $\begin{bmatrix} 1&0&0&0\\ 0&k&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix} \leftarrow \det = k$

- Add k times row 2 to row 4 is realized by $\begin{bmatrix} 1&0&0&0\\ 0&1&0&k\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix} \leftarrow \det = 1$

So now consider computing the determinant of $\begin{bmatrix} 4 & 4 & 4 \\ 2 & 3 & 3 \\ -4 & -5 & -1 \end{bmatrix}$:

We can now reduce:

$$\begin{bmatrix} 4 & 4 & 4 \\ 2 & 3 & 3 \\ -4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{Swap } [1] \,\&\, [2]} \begin{bmatrix} 2 & 3 & 3 \\ 4 & 4 & 4 \\ -4 & -5 & -1 \end{bmatrix} \xrightarrow[{[3] \to [3]+2[1]}]{[2] \to [2]-2[1]} \begin{bmatrix} 2 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{[2] \to -\frac{1}{2}[2]} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{[3] \to [3]-[2]} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

So now

$$\underset{\det = 8}{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}} = \underset{\det = 1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\det = -1/2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\det = 1}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\det = 1}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\det = -1}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 4 & 4 & 4 \\ 2 & 3 & 3 \\ -4 & -5 & -1 \end{bmatrix}$$

So $8 = \frac{1}{2}\det(A) \Rightarrow \det A = 16$

Check: $\begin{vmatrix} 4 & 4 & 4 \\ 2 & 3 & 3 \\ -4 & -5 & -1 \end{vmatrix} = 4\begin{vmatrix} 3 & 3 \\ -5 & -1 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} = 48 - 40 + 8 = 16$

We can just record the determinant of the corresponding elementary matrices as we go along:

$$\begin{vmatrix} 4 & 4 & 4 \\ 2 & 3 & 3 \\ -4 & -5 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & 3 & 3 \\ 4 & 4 & 4 \\ -4 & -5 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 5 \end{vmatrix} = -(-2)\begin{vmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{vmatrix} = 2\cdot(2\cdot 1\cdot 4) = 16 \quad \checkmark.$$

And of course if you ever come across a row of all zeros, you know your determinant must be 0.