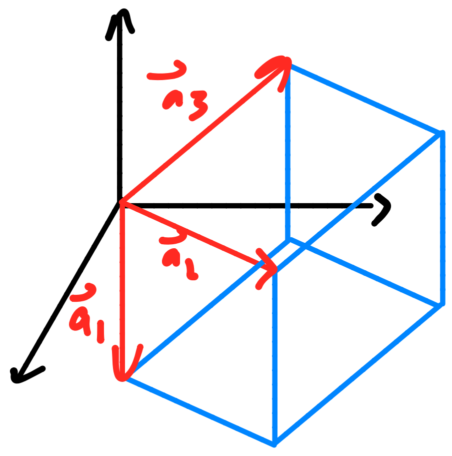
Math 369: Day 10

Determinants

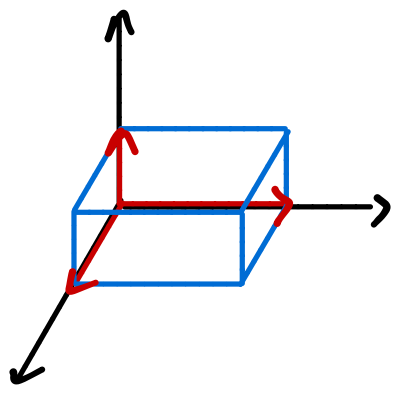
I don't want to overemphasize determinants too much b/c they're essentially uncomputable, but they do have useful theoretical properties,

so it's useful to say a few words about them.

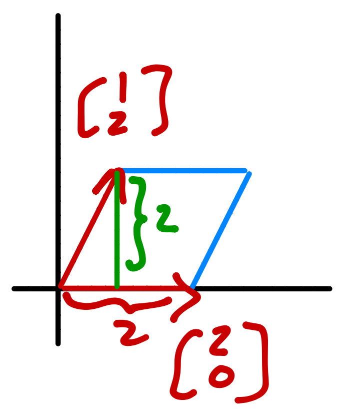
If $A \in Mat(n,n)$, then $\det(A)$ is exactly the (signed) n-dim'l volume of the box determined by the columns of A.



Ex: If $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $\det(A) = abc$

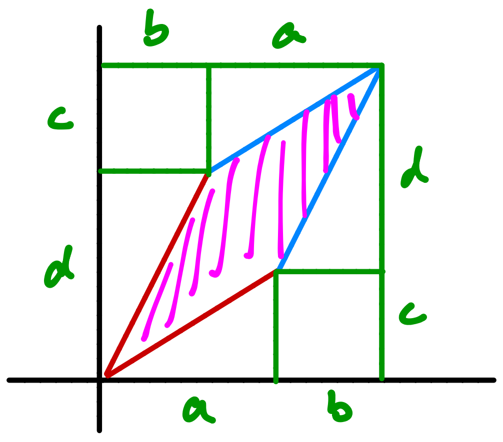


Ex: $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Then $\det(A) = 4$



Area $= (\text{base})(\text{height}) = 2 \cdot 2 = 4$

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$



Area $= (a+b)(c+d) - \left(\frac{1}{2}ac + \frac{1}{2}bd + bc + \frac{1}{2}bd + \frac{1}{2}ac + bc\right)$

$= \cancel{ac} + \cancel{bc} + ad + \cancel{bd} - \cancel{ac} - \cancel{bd} - \cancel{2}bc$

$= ad - bc$

If we interchange rows, that changes the sign of the determinant.

There is exactly one function w/ these properties, the determinant function, & it follows that $\det(AB) = \det(A)\det(B)$.

Moreover, TFAE:
- $\det(A) = 0$
- The image of the unit cube has dimension less than n
- A is not invertible.

In general, if $A \in Mat(n,n)$ & $A=(a_{ij})$ & A_{ij} is the submatrix given by deleting i^{th} row & j^{th} column, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

For any i.

Ex: $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (-1)^{1+1} 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + (-1)^{1+2} 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (-1)^{1+3} 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 7\cdot 6) + 3(4\cdot 8 - 7\cdot 5)$

$= -3 + 12 - 9 = 0$

expanding on first row

or

$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (-1)^{2+1} \cdot 4 \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + (-1)^{2+2} \cdot 5 \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-1)^{2+3} \cdot 6 \cdot \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -4(2\cdot 9 - 8\cdot 3) + 5(1\cdot 9 - 7\cdot 3) - 6(1\cdot 8 - 7\cdot 2)$

$= 24 - 60 + 36 = 0$

expanding on 2nd row

In general, some rows will be better choices to expand on than others. For example, if some row has lots of zeros, that's often a good row to expand on.