Math 369 Final Exam Practice Problem Solutions

1. Find all solutions of the system of equations

$$\begin{array}{rcrcrcrcl} x & - & y & - & 2z & - & 5w & = & 1 \\ 4x & - & 3y & - & 6z & - & 15w & = & 2 \\ -2x & - & 4y & - & 7z & - & 18w & = & 8. \end{array}$$

Simplify your answer as much as possible.

Answer: This system corresponds to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -5 & 1 \\ 4 & -3 & -6 & -15 & 2 \\ -2 & -4 & -7 & -18 & 8 \end{array}\right].$$

So now we row-reduce. Subtract 4 times row 1 from row 2 and add twice row 1 to row 3:

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -5 & 1 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & -6 & -11 & -28 & 10 \end{array}\right].$$

Next add 6 times row 2 to row 3:

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -5 & 1 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right].$$

The last row corresponds to the equation $z + 2w = -2$, or $z = -2 - 2w$. Since the middle row corresponds to $y + 2z + 5w = -2$, we can re-write as $y = -2 - 2z - 5w$ and then plug in $z = -2 - 2w$ to get

$$y = -2 - 2z - 5w = -2 - 2(-2 - 2w) - 5w = 2 - w.$$

Finally, then, the first row corresponds to $x - y - 2z - 5w = 1$ or $x = 1 + y + 2z + 5w$, which we can plug $z = -2 - 2w$ and $y = 2 - w$ into:

$$x = 1 + y + 2z + 5w = 1 + (2 - w) + 2(-2 - 2w) + 5w = -1.$$

Hence, solutions of the system of equations are all x, y, z, w with

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 2 - w \\ -2 - 2w \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

2. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 2x + y \\ 3x + y \end{bmatrix}.$$

Determine the subspace $S = \{\vec{u} \in \mathbb{R}^2 : T(\vec{u}) = \vec{0}\}$.

Answer: By definition, we want to find the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that solve $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In other words, since

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

is the matrix for T, we're just trying to find the nullspace of A. So row-reduce the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right].$$

Subtracting twice row 1 from row 2 and three times row 1 from row 3 yields

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right],$$

and then subtracting twice row 2 from row 3 yields

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore, the reduced system of equations is

$$\begin{aligned} x + y &= 0 \\ -y &= 0, \end{aligned}$$

so in fact $y = 0$ and $x = 0$. Therefore, the nullspace consists only of the zero vector:

$$\text{null}(T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

3. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ x \end{bmatrix} \right\}.$$

   (a) Find a value for x which makes $\mathcal{B}$ linearly dependent and prove that the result really is linearly dependent.

   Answer: If $x = 6$, then we see that

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

   so this set is linearly dependent.

(b) Find a value for x which makes $\mathcal{B}$ linearly *independent* and prove that the result really is linearly independent.

Answer: It turns out that any choice *other than* $x = 6$ will work, so we could choose, say, $x = 0$. Suppose there were a linear combination giving zero:

$$a \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This corresponds to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 2 & 4 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right].$$

Now row-reduce: subtract twice row 1 from row 2 and four times row 1 from row 3:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -10 & -16 & 0 \end{array}\right].$$

Subtract 5/2 times row 2 from row 3:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right].$$

This gives the system of equations

$$\begin{aligned} a + 3b + 4c &= 0 \\ -4b - 4c &= 0 \\ -6c &= 0 \end{aligned}$$

The third equation says $c = 0$; plugging that into the second equation gives $b = 0$, and then plugging both into the first equation gives $a = 0$. So we see that the *only* linear combination of the vectors giving the zero vector is the trivial linear combination, and hence the three vectors are linearly independent.

4. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} \right\}$.

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Answer: There are many ways to prove that $\mathcal{B}$ is a basis. Since $\mathbb{R}^3$ is 3-dimensional and $\mathcal{B}$ has three elements, it suffices to show that the three vectors are linearly independent. Since we will need it in part (b) anyway, we demonstrate this by showing that the matrix A whose columns are the elements of $\mathcal{B}$ is invertible. We can compute the inverse by reducing the super-augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{array}\right].$$

First, subtract row 1 from row 2 and subtract three times row 1 from row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 2 & -9 & -3 & 0 & 1 \end{array}\right].$$

Next, subtract twice row 2 from row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right].$$

Finally, subtract 5 times row 3 from row 1, and add 5 times row 3 to row 2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 10 & -5 \\ 0 & 1 & 0 & -6 & -9 & 5 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right].$$

Therefore,

$$A^{-1} = \begin{bmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{bmatrix}$$

so A is invertible and therefore the elements of $\mathcal{B}$ are linearly independent. Since any set of three linearly indpendent vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$, this shows that $\mathcal{B}$ is a basis.

(b) Write $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the elements of $\mathcal{B}$.

Answer: The coefficients in the linear combination will just be the entries in the coordinate vector $[\vec{u}]_{\mathcal{B}}$ for $\vec{u}$ with respect to the $\mathcal{B}$ basis. If $\mathcal{E}$ is the standard basis for $\mathbb{R}^3$, then the process for computing the change-of-basis matrix $[M]_{\mathcal{E}\to\mathcal{B}}$ is exactly the process we just did for computing A^{-1}. Said another way, $[M]_{\mathcal{E}\to\mathcal{B}} = A^{-1}$. Therefore, We have

$$[\vec{u}]_{\mathcal{B}} = [M]_{\mathcal{E}\to\mathcal{B}}[\vec{u}]_{\mathcal{E}} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

In other words, this says we can write $\vec{u}$ as a linear combination

$$\vec{u} = 1\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix},$$

which checks out.

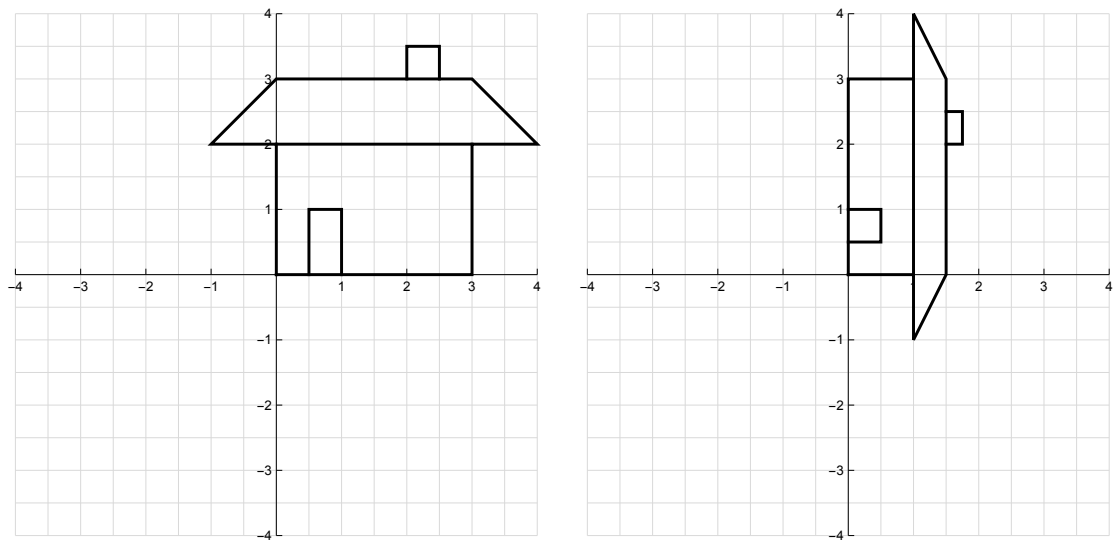
5. Alice and Bob are each given the same 7×11 matrix B. Alice computes that the nullspace of B is 3-dimensional, while Bob computes that the column space of B is 8-dimensional. Can they both be right? Why or why not?

Answer: No, this is not possible. Although the rank-nullity theorem says that

$$\#(\text{columns of } B) = \dim \operatorname{null}(B) + \dim \operatorname{col}(B)$$

and these numbers don't violate that, there is no way that the column space of B can be 8-dimensional. Since B has only 7 rows, the columns of B are 7-dimensional vectors, and there is no way that a collection of 7-dimensional vectors can span an 8-dimensional space.

6. Shown below left is a picture before applying an unknown matrix C, and below right is the result after applying the matrix. What is the absolute value of $\det(C)$? (You do *not* need to determine the matrix C to solve this problem, though of course that is one approach.)



Answer: With some work, you can show that the matrix $C = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}$, so $|\det(C)| = |0 - 1 \cdot \frac{1}{2}| = |-\frac{1}{2}| = \frac{1}{2}$.

However, you could also just figure this out directly from the picture. Remember that the determinant of the matrix is supposed to be the volume of the image of a unit box under the action of the matrix. Since each square in the original picture gets sent to a rectangle of the same height and half the width, the image of each box will have area equal to half the original area (indeed, the area of the new house is half the area of the old house). So the absolute value of the determinant must be $1/2$.

7. Let $\mathcal{C}^0([-1,1])$ be the vector space of continuous functions on the closed interval $[-1,1]$. Let $\mathcal{E}$ be the set of *even* continuous functions on $[-1,1]$; in other words,

$$\mathcal{E} = \{f \in \mathcal{C}^0([-1,1]) : f(x) = f(-x) \text{ for all } x \in [-1,1]\}.$$

Prove that $\mathcal{E}$ is a subspace of $\mathcal{C}^0([-1,1])$.

Proof. As usual, we have to show that $\mathcal{E}$ satisfies the three subspace axioms.

First, if $z(x) = 0$ is the zero function, then $z(x) = 0 = z(-x)$ for all $x \in [-1, 1]$, so $z \in \mathcal{E}$.

Next, if $f, g \in \mathcal{E}$, then we know that $f(x) = f(-x)$ and $g(x) = g(-x)$ for all $x \in [-1, 1]$. Therefore, for any $x \in [-1, 1]$, we have that

$$(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x),$$

so $f + g \in \mathcal{E}$.

Finally, if $h \in \mathcal{E}$ and $\lambda \in \mathbb{R}$, then we know that $h(x) = h(-x)$ for all x, so

$$(\lambda h)(x) = \lambda h(x) = \lambda h(-x) = (\lambda h)(-x)$$

for all $x \in [-1, 1]$, so we see that $\lambda h \in \mathcal{E}$.

Therefore, since it satisfies the three conditions, we conclude that $\mathcal{E}$ is a subspace of $\mathcal{C}^0([-1, 1])$. $\square$

8. Suppose that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for the vector space V. If $c_1, \ldots, c_n$ are non-zero numbers, prove that $\{c_1\vec{v}_1, \ldots, c_n\vec{v}_n\}$ is *also* a basis for V.

Proof. Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V, it is clear that V is n-dimensional. Since $\{c_1\vec{v}_1, \ldots, c_n\vec{v}_n\}$ also contains n elements, and since any linearly independent n element subset of an n-dimensional vector space must be a basis, it suffices to show that $\{c_1\vec{v}_1, \ldots, c_n\vec{v}_n\}$ is linearly independent.

So now suppose there were coefficients $a_1, \ldots, a_n$ so that

$$a_1(c_1\vec{v}_1) + a_2(c_2\vec{v}_2) + \cdots + a_n(c_n\vec{v}_n) = \vec{0}.$$

Then we can re-group coefficients and see that

$$(a_1c_1)\vec{v}_1 + (a_2c_2)\vec{v}_2 + \cdots + (a_nc_n)\vec{v}_n = \vec{0}.$$

But now this is a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_n$ giving the zero vector. Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis, this can only happen if the coefficients $a_1c_1, \ldots, a_nc_n$ are all zero. But since none of the c_i are zero, this means all the a_i are zero.

So this says that the only linear combination of $c_1\vec{v}_1, \ldots, c_n\vec{v}_n$ giving the zero vector is the trivial combination, which implies that the $c_1\vec{v}_1, \ldots, c_n\vec{v}_n$ are linearly independent, and hence form a basis. $\square$

9. The 3×3 matrix

$$C = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 2 & -2 \\ 2 & 0 & 0 \end{bmatrix}$$

has 1 and 2 as eigenvalues (you do not need to prove this, just take it as given).

(a) Find a basis for the eigenspace $V_1 = \{\vec{v} \in \mathbb{R}^3 : B\vec{v} = \vec{v}\}$.
Answer: We're solving $(C - I)\vec{x} = \vec{0}$, so form the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

and row-reduce. Subtract row 1 from both row 2 and row 3:

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then our system of equations is

$$\begin{aligned} 2x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0, \end{aligned}$$

so we see that $x_2 = x_3$ and $x_1 = \frac{1}{2}x_3$. Therefore, all eigenvectors are of the form $x_3 \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$, so $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for V_1.

(b) Find a basis for the eigenspace $V_2 = \{\vec{v} \in \mathbb{R}^3 : B\vec{v} = 2\vec{v}\}$.
Answer: We're solving $(C - 2I)\vec{x} = \vec{0}$, so form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right]$$

and row-reduce. Subtract twice row 1 from both row 2 and row 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we just have the single equation $x_1 - x_3 = 0$, or $x_1 = x_3$. Therefore, all eigenvectors are of the form

$$\begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for V_2.

10. Let $\mathcal{P}_1$ be the vector space of polynomials in the variable x of degree ≤ 1.

(a) Show that $\{1, x\}$ is a basis for $\mathcal{P}_1$.

Proof. Any element of $\mathcal{P}_1$ is of the form $a + bx$, which is a linear combination of 1 and x, so this is definitely a spanning set. Moreover, if a linear combination

$$a(1) + b(x) = 0,$$

then we see that $a = 0$ and $b = 0$, so only the trivial linear combination gives the zero element of $\mathcal{P}_1$, so this set is linearly independent. The set is a linearly independent spanning set, so it is a basis. $\square$

(b) We can define an inner product on $\mathcal{P}_1$ by letting

$$\langle p, q\rangle = p(0)q(0) + p(1)q(1)$$

for any linear polynomials p and q (this is a version of the evaluation inner product we've talked about before). Apply the Gram-Schmidt process to the basis from part (a) to find an *orthonormal* basis for $\mathcal{P}_1$.

Answer: Let $p(x) = 1$ and let $q(x) = x$. Then

$$\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{p(0)p(0) + p(1)p(1)} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

so we get

$$v_1 = \frac{1}{\|p\|}p = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}}.$$

Next,

$$\langle q, v_1\rangle = q(0)v_1(0) + q(1)v_1(1) = 0 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

so

$$w_2 = q - \langle q, v_1\rangle v_1 = x - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = x - \frac{1}{2}.$$

Finally,

$$\begin{aligned}\|w_2\| = \sqrt{\langle w_2, w_2\rangle} &= \sqrt{w_2(0)w_2(0) + w_2(1)w_2(1)} \\ &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}},\end{aligned}$$

so

$$v_2 = \frac{1}{\|w_2\|}w_2 = \frac{1}{1/\sqrt{2}}\left(x - \frac{1}{2}\right) = \sqrt{2}\left(x - \frac{1}{2}\right) = \sqrt{2}x - \frac{\sqrt{2}}{2} = \sqrt{2}x - \frac{1}{\sqrt{2}}.$$

Therefore, $\left\{\frac{1}{\sqrt{2}}, \sqrt{2}x - \frac{1}{\sqrt{2}}\right\}$ is an orthonormal basis for $\mathcal{P}_1$ with respect to this evaluation inner product.

11. An astronaut on a space station is out making repairs in a small shuttle when the shuttle's thrusters mysteriously conk out. With no thrusters, the astronaut is stranded and needs to be rescued. The stranded shuttle is at a position 10 meters east, 100 meters north, and 50 meters up from the station's docking bay; if we choose coordinates with the origin at the docking bay, the stranded shuttle is at the point $(10, 100, 50)$.

There is a backup shuttle, but one of its thrusters is malfunctioning, so it is only capable of moving in two directions: by engaging the first thruster the backup shuttle can go in the direction of the unit vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and by engaging the second it can go in the direction of the unit vector $\vec{e}_2 = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$. Of course, by engaging both thrusters simultaneously and with different intensities, the backup shuttle can go in any direction of the form $a\vec{e}_1 + b\vec{e}_2$.

(a) Given these constraints, what is the closest point to the stranded shuttle that the backup shuttle can get to? You should use the usual notion of distance in $\mathbb{R}^3$, which corresponds to the norm coming from the standard dot product. (*Hint: first think about how to translate this into a linear algebra question.*)

Answer: The backup shuttle is only able to move in the 2-dimensional plane W spanned by $\vec{e}_1$ and $\vec{e}_2$. Therefore, the closest point to the stranded shuttle that it can reach is the projection of the vector $\vec{v} = \begin{bmatrix} 10 \\ 100 \\ 50 \end{bmatrix}$ onto W.

Now, notice that $\{\vec{e}_1, \vec{e}_2\}$ is already an orthonormal basis for W:

$$\begin{aligned}
\vec{e}_1 \cdot \vec{e}_1 &= 1^2 + 0^2 + 0^2 = 1 \\
\vec{e}_2 \cdot \vec{e}_2 &= 0^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \\
\vec{e}_1 \cdot \vec{e}_2 &= 1 \times 0 + 0 \times \frac{3}{5} + 0 \times \frac{4}{5} = 0.
\end{aligned}$$

Therefore, the projection of $\vec{v}$ onto W is simply

$$\begin{aligned}
\operatorname{proj}_W \vec{v} = (\vec{v} \cdot \vec{e}_1)\vec{e}_1 + (\vec{v} \cdot \vec{e}_2)\vec{e}_2 &= 10 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(100 \times \frac{3}{5} + 50 \times \frac{4}{5}\right) \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} \\
&= \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + 100 \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} \\
&= \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \\ 80 \end{bmatrix} \\
&= \begin{bmatrix} 10 \\ 60 \\ 80 \end{bmatrix}.
\end{aligned}$$

So the closest point that the backup shuttle can reach is the point $(10, 60, 80)$.

(b) How long of a tether will need to be packed in the backup shuttle from Question 10 in order to rescue the stranded astronaut?

Answer: Of course, this is just the distance from $(10, 60, 80)$ to $(10, 100, 50)$, which is

$$\sqrt{(10-10)^2 + (60-100)^2 + (80-50)^2} = \sqrt{0^2 + 40^2 + 30^2} = \sqrt{50^2} = 50.$$

That is to say, the backup shuttle needs to pack a 50 meter tether in order to be able to rescue the stranded astronaut.

12. Consider the second-order differential equation

$$y'' - 3y' - 4y = 0.$$

Introduce the auxiliary functions $y_1 = y$ and $y_2 = y'$.

(a) Re-write the given second-order equation as a system of two first-order equations in y_1 and y_2.
Answer: We know that

$$y_1' = y' = y_2$$

and that

$$y_2' = y'' = 4y + 3y' = 4y_1 + 3y_2,$$

so we can translate the second-order equation into the first-order system

$$\begin{aligned} y_1' &= 0y_1 + 1y_2 \\ y_2' &= 4y_1 + 3y_2. \end{aligned}$$

(b) Use this reformulation to solve the original equation.
Answer: We solve the differential equation

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

in the usual way: we diagonalize $A = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}$, getting $A = PDP^{-1}$, then solve the easier system $\begin{bmatrix} u_1' \\ u_1' \end{bmatrix} = D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and finally get $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
First of all, to diagonalize A we need to know the eigenvalues, which are the roots of the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = -\lambda(3-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4),$$

so the eigenvalues are $\lambda = -1$ and $\lambda = 4$. Now we find the corresponding eigenvectors:
$\lambda = -1$**:** We need to solve $(A + I)\vec{x} = \vec{0}$, so we form the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

and row-reduce. Subtracting 4 times row 1 from row 2 yields

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so the equation is just $x_1 + x_2 = 0$, or $x_1 = -x_2$. Therefore, the corresponding eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\lambda = 4$**:** We need to solve $(A - 4I)\vec{x} = \vec{0}$, so we form the augmented matrix

$$\left[\begin{array}{cc|c} -4 & 1 & 0 \\ 4 & -1 & 0 \end{array}\right]$$

and row-reduce. Adding row 1 to row 2 yields

$$\left[\begin{array}{cc|c} -4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so the equation is just $-4x_1 + x_2 = 0$, or $x_2 = 4x_1$. Therefore, the corresponding eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Hence, our matrices P and D are

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Now, solving

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ 4u_2 \end{bmatrix}$$

is easy: $u_1(x) = C_1 e^{-x}$ and $u_2(x) = C_2 e^{4x}$.

Therefore,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C_1 e^{-x} \\ C_2 e^{4x} \end{bmatrix} = \begin{bmatrix} -C_1 e^{-x} + C_2 e^{4x} \\ C_1 e^{-x} + 4C_2 e^{4x} \end{bmatrix}.$$

Finally, then, we see that

$$y = y_1 = -C_1 e^{-x} + C_2 e^{4x}$$

is a solution to the original equation.

We can check this by computing

$$\begin{aligned} y' &= C_1 e^{-x} + 4C_2 e^{4x} \\ y'' &= -C_1 e^{-x} + 16C_2 e^{4x} \end{aligned}$$

and then plugging y, y', and y'' into the left hand side of the original equation and verifying that we get zero:

$$\begin{aligned} y'' - 3y' - 4y &= \left(-C_1 e^{-x} + 16C_2 e^{4x}\right) - 3\left(C_1 e^{-x} + 4C_2 e^{4x}\right) - 4\left(-C_1 e^{-x} + C_2 e^{4x}\right) \\ &= (-C_1 - 3C_1 + 4C_1)e^{-x} + (16C_2 - 12C_2 - 4C_2)e^{4x} = 0. \end{aligned}$$