Math 369 Exam #3 Practice Problems

1. Diagonalize the matrix
$$A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$$
In other words, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$.

2. Let A be an $n \times n$ matrix.
   (a) Let $\lambda \in \mathbb{R}$ be a number. Give the definition of the eigenspace V_λ of A associated to λ.
   (b) Show that $\vec{0} \in V_\lambda$ (this is true regardles of what λ is).
   (c) Suppose that $\lambda \neq \tau$, but that $\vec{v} \in V_\lambda$ and $\vec{v} \in V_\tau$. Show that this means $\vec{v} = \vec{0}$.

3. Let V be a vector space with an inner product. Let $v \in V$ be some particular vector and define
$$W = \{w \in V : \langle w, v \rangle = 0\}.$$
Prove that W is a subspace of V.

4. Consider $\mathbb{R}^3$ with the slightly unusual inner product
$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{6}u_1v_1 + \frac{1}{8}u_2v_2 + \frac{1}{27}u_3v_3.$$
Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.
   (a) Show that $\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal set with respect to this weird inner product.
   (b) Let $\vec{u}_3 = \begin{bmatrix} 0 \\ 8 \\ 18 \end{bmatrix}$. Apply the Gram-Schmidt procedure to the set $\{\vec{v}_1, \vec{v}_2, \vec{u}_3\}$ to get an orthonormal set. (*Hint:* you already know from part (a) that $\{\vec{v}_1, \vec{v}_2\}$ is orthonormal, so $\vec{u}_3$ is the only vector that needs adjustment.)

5. Solve the system of differential equations
$$\begin{aligned} y_1' &= -2y_1 + 2y_2 \\ y_2' &= y_1 - y_2. \end{aligned}$$

6. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.
   (a) Every invertible matrix can be diagonalized.
   (b) Every diagonalizable matrix is invertible.
   (c) If the matrix A is not invertible, then 0 is an eigenvalue of A.

7. Let
$$B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix}$$
   (a) Find an orthonormal basis (using the usual dot product) for the column space of B.
   (b) Use your answer from part (a) to compute the orthogonal projection of $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ onto $\mathrm{col}(B)$.