Math 369 Final Exam Practice Problems

1. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 2x+y \\ 3x+y \end{pmatrix}.$$

   What is the nullspace of T?

2. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ x \end{pmatrix} \right\}.$$

   (a) Find a value for x which makes $\mathcal{B}$ linearly dependent and prove that the result really is linearly dependent.

   (b) Find a value for x which makes $\mathcal{B}$ linearly *independent* and prove that the result really is linearly independent.

3. Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} -4 & -2 & -3 \\ 2 & 1 & 6 \\ 1 & 2 & 0 \end{pmatrix}.$$

4. Let $F : V \to V$ be a linear transformation.

   (a) Let $\lambda \in \mathbb{R}$ be a number. Give the definition of the eigenspace V_λ of F associated to λ.

   (b) Show that $\vec{0} \in V_\lambda$ (this is true regardles of what λ is).

   (c) Suppose that $\lambda \neq \tau$, but that $\vec{v} \in V_\lambda$ and $\vec{v} \in V_\tau$. Prove that this means $\vec{v} = \vec{0}$.

5. Let V be a vector space with an inner product. Let $\vec{v} \in V$ be some particular vector and define

$$W = \{\vec{w} \in V : \langle \vec{w}, \vec{v} \rangle = 0\}.$$

   Prove that W is a subspace of V.

6. Alice and Bob are each given the same 7×11 matrix B. Alice computes that the nullspace of B is 3-dimensional, while Bob computes that the column space of B is 8-dimensional. Can they both be right? Why or why not?

7. Consider $\mathbb{R}^3$ with the slightly unusual inner product

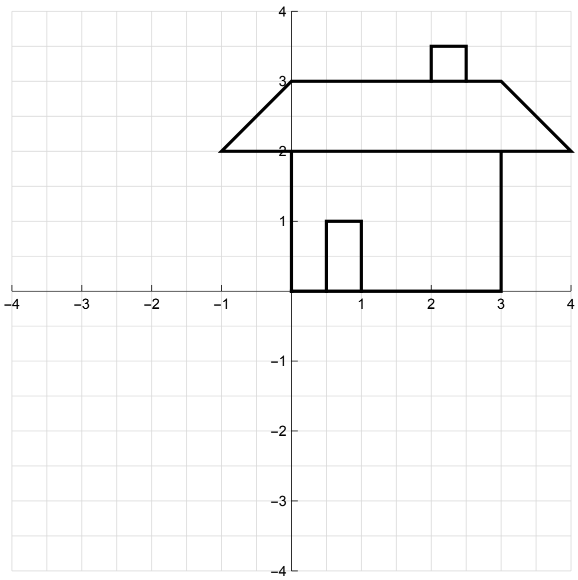
$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{6}u_1v_1 + \frac{1}{8}u_2v_2 + \frac{1}{27}u_3v_3.$$

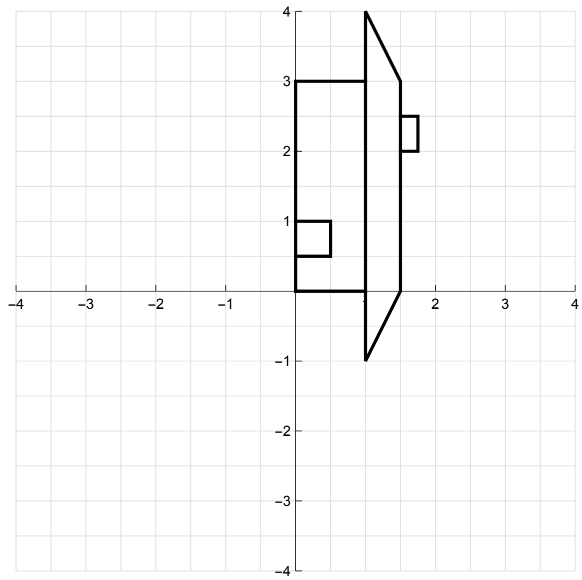
   Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

   (a) Show that $\{\vec{e}_1, \vec{e}_2\}$ is an orthonormal set with respect to this weird inner product.

   (b) Let $\vec{v}_3 = \begin{pmatrix} 0 \\ 8 \\ 18 \end{pmatrix}$. Apply the Gram-Schmidt prodcedure to the set $\{\vec{e}_1, \vec{e}_2, \vec{v}_3\}$ to get an orthonormal set. (*Hint:* you already know from part (a) that $\{\vec{e}_1, \vec{e}_2\}$ is orthonormal, so $\vec{v}_3$ is the only vector that needs adjustment.)

8. Shown below left is a picture before applying an unknown matrix C, and below right is the result after applying the matrix. What is the absolute value of $\det(C)$? (You do *not* need to determine the matrix C to solve this problem, though of course that is one approach.)





9. Prove or disprove the following claim: If A and B are $n \times n$ matrices and A is invertible, then $\det(ABA^{-1}) = \det(B)$.

10. Let $\mathcal{C}^0([-1,1])$ be the vector space of continuous functions on the closed interval $[-1,1]$. Let $\mathcal{E}$ be the set of *even* continuous functions on $[-1,1]$; in other words,

$$\mathcal{E} = \{f \in \mathcal{C}^0([-1,1]) : f(x) = f(-x) \text{ for all } x \in [-1,1]\}.$$

Prove that $\mathcal{E}$ is a subspace of $\mathcal{C}^0([-1,1])$.