Math 336 HW #6

Due 5:00 PM Tuesday, March 30

Presentation:

1. Prove that if $f : K \to L$ is a Δ-map, then the following diagram commutes (notation as in class):

$$\begin{array}{ccc} H_n^{\Delta}(K) & \xrightarrow{f_*} & H_n^{\Delta}(L) \\ \downarrow \phi_* & & \downarrow \phi_* \\ H_n(|K|) & \xrightarrow{|f|_*} & H_n(|L|) \end{array}$$

Problems:

1. Hatcher p. 133 #27
2. Hatcher p. 133 #29. By now, you can definitely use simplicial homology!
3. Let $I \subset S^2$ be the image of an embedding of $[0, 1]$, and let x be in the image of $(0, 1)$. Prove that the inclusion $i_* : H_*(S^2 \setminus x, I \setminus x) \to (S^2, I)$ is not an isomorphism.
4. To prove the homotopy invariance property of homology (i.e. if $f, g : X \to Y$ are homotopic, then they induce the same map on homology), we constructed a chain homotopy between $f_\#$ and $g_\#$ using the prism operator P. To better understand one part of the proof of excision, let's construct a different chain homotopy in a similar manner to the construction of the subdivision chain map S.
   (a) Define an operation T^{Δ} on the identity map $\delta_n : \Delta^n \to \Delta^n$ whose image is a chain with image in $\Delta^n \times I$ inductively as follows. Let $i_j : \Delta^n \to \Delta_n \times I$ for $j = 0, 1$ be the inclusion into $\Delta^n \times \{j\}$. Let v be point $q \times \{\frac{1}{2}\} \in \Delta^n \times I$, where q is the barycenter of Δ^n. Then make the following inductive definition:
   $$T^{\Delta}\delta_n = c_v(i_1\delta_n - i_0\delta_n - T^{\Delta}\partial\delta_n).$$
   Prove that T^{Δ} acts like a chain homotopy operator between i_0 and i_1 on the simplex δ_n (that is, it satisfies the same equation when applied to δ_n).
   (b) Transfer T^{Δ} to a map from $C_n(X)$ to $C_{n+1}(Y)$ and prove that it is a chain homotopy between $f_\#$ and $g_\#$.
5. Let K be a finite Δ-complex with c_i i-simplices, and define the *Euler characteristic of* K to be:
   $$\chi(K) = \sum_{i=0}^{\infty}(-1)^i c_i.$$
   Prove that if $|K|$ is homeomorphic to $|L|$, then $\chi(K) = \chi(L)$.

HINT: Think of c_i as the rank of the free abelian group $\Delta_i(K)$ and show that $\chi(K) = \sum_{i=0}^{\infty}(-1)^i \text{rank } H_i(|K|)$. Note that the rank of a free abelian group is simply the rank of its free part (e.g. rank $(\mathbb{Z}^2 \oplus \mathbb{Z}_3) = 2$). You'll want to generalize the Rank-Nullity theorem to finitely generated abelian groups.

NOTE: This is the "real" reason that Euler's Formula $V - E + F = 2$ is true (where V, E, and F are the number of vertices, edges, and faces, respectively, of a polyhedron).