Math 336 HW #3

Due 5:00 PM Tuesday, February 16

Presentations:

1. In the proof that a short exact sequence of chain complex yields a long exact sequence in homology, we defined a map $\partial_* : H_n(C) \to H_{n-1}(A)$. Show that this map is well-defined, i.e. that it doesn't depend on the choice of β and that $\partial_*[\alpha] = \partial_*[\alpha + \partial_C w]$.

2. Again in the proof that short exact sequence of chain complex yields a long exact sequence in homology, we left out the verification of exactness at $H_n(B)$. Do it.

Problems:

1. Before doing this problem, you should convince yourself of the following facts about the maps induced by Δ-maps (you do not need to write down the proofs):

   (a) If $f : K \to K'$ and $g : K' \to K''$ are Δ-maps, then so is $f \circ g$ and, further, $(f \circ g)_\# = f_\# \circ g_\#$.

   (b) If $i : K \to K$ is the identity Δ-map, then $i_\#$ is the identity map on the simplicial chain complex $\Delta_*(K)$.

   Now prove that if $A \subset K$ is a subcomplex and $r : K \to A$ is a Δ-map such that $r \circ i$ is the identity on K, then the map induced by inclusion $i_* : H_n^\Delta(A) \to H_n^\Delta(K)$ is injective.

2. (a) Hatcher p. 132 #15. For the second part, there's really nothing for you to do except "notice" the conclusion for yourselves — and this really tells you important information about the meaning and use of the relative homology!

   (b) If $\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to C \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}$ is exact, where "$\times 2$" is the map $f(n) = 2n$, then find the group C.

3. Hatcher p. 132 #17(b). For this problem, you should still be using simplicial homology, so you'll have to find Δ-complexes for the surface of genus 2 that contain A and B as subcomplexes.

4. Prove the (incredibly useful) 5-lemma: if the top and bottom rows of the following diagram are exact, and if α, β, δ, and ϵ are isomorphisms, then prove that γ is also an isomorphism.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

5. (a) Let $K^{(1)}$ be the union of 1- and 0-simplices of a Δ-complex K, and assume that $|K^{(1)}|$ is connected. Prove that $H_0^\Delta(K) \simeq \mathbb{Z}$, generated by the homology class $[x]$ for some (any!) 0-simplex x.

(b) Let K be a Δ-complex and let CK be its cone, as defined in the last problem set. Let $K^{(1)}$ be the union of 1- and 0-simplices of K, and assume that $|K^{(1)}|$ is connected. Compute $H_n^{\Delta}(CK, K)$ for all n.