Math 3200: Days 17-18

Problems to be presented: 4.18, 4.34, 4.48

Thm 1: $\sqrt{2}$ is irrational.

How in the world would we try to prove this?

...It's not so clear that either a direct proof or a proof by contrapositive will work, esp. since there's no obvious implication here.

So what are other ways to tackle it?

It turns out that a good strategy is a new approach: proof by contradiction.

Here's the idea: if we want to prove a statement P by contradiction, then we assume that $\sim P$ is true and then argue that $\sim P$ implies some contradiction C: $(\sim P) \Rightarrow C$ (recall that a contradiction is a statement which is false regardless of the truth values of any of its component statements).

But then $(\sim P) \Rightarrow C$ is a true statement and C is a false statement. Looking at the truth table, this must mean $\sim P$ is false... and hence P is true!

P	C	$\sim P$	$(\sim P) \Rightarrow C$
T	F	F	T
F	F	T	F

So that's the idea: to prove P by contradiction, assume the negation of P and show it leads to a contradiction.

Before tackling Thm 1, let's warm up w/ a proof by contradiction w/ fewer moving parts:

Prop: There is no smallest positive rational number.

Proof: Suppose, to the contrary, that there were a smallest positive rational number r. Then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

But now $\frac{r}{2} = \frac{a}{2b}$ is rational & $0 < \frac{r}{2} < r$, so $\frac{r}{2}$ is a smaller positive rational than r, so r is not the smallest.

Therefore, the assumption that there is a smallest positive rational number implied the statement "r is the smallest positive rational and r is not the smallest positive rational", a clear contradiction. Therefore, since the assumption that there is a smallest positive rational led to a contradiction, we can only conclude that there is not a smallest positive rational number. $\square$

In general, what typically happens is that assuming $\sim P$ ultimately implies a statement which is the negation of the assumption, the negation of a known fact, or the negation of another consequence of $\sim P$. In any case, the contradiction C implied by the assumption $\sim P$ is usually of the form $C = Q \wedge (\sim Q)$ for some statement Q.

Restatement of Thm 1: $\sqrt{2}$ is irrational.

Proof of Thm 1: Assume, for the sake of deriving a contradiction, that $\sqrt{2}$ is rational. Then by definition this means

$$\sqrt{2} = \frac{a}{b} \quad \text{for some } a, b \in \mathbb{Z} \text{ so that } b \neq 0 \text{ and } a \,\&\, b \text{ have no common divisors (i.e., } \tfrac{a}{b} \text{ is in lowest terms)}$$

Well, if $\sqrt{2} = \frac{a}{b}$, then $2 = \sqrt{2}^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$ and we see that a^2 is even. But since a & a^2 have the same prime factors, if a^2 is divisible by 2 then so is a. Thus, $a = 2k$ for some $k \in \mathbb{Z}$.

Now, return to the equation $a^2 = 2b^2$. Substituting in $a = 2k$, we see that

$$2b^2 = a^2 = (2k)^2 = 4k^2$$

so $b^2 = 2k^2$. But this means b^2 is even and hence b is also even.

But that means a & b are both divisible by 2, directly contradicting the fact that a & b are relatively prime.

Since the assumption that $\sqrt{2}$ was rational led to a contradiction, we've proved that $\sqrt{2}$ is, in fact, irrational. $\square$

(If there's time, do as much or as little as you like of the following)

Thm 2: There are infinitely many primes.