Math 3200: Day 9

Proofs:

Q: What is a mathematical proof?

A: A proof of a statement (theorem, lemma, proposition, etc.) is an argument that convinces your audience that the statement is true.

Notice that this depends on your audience, as well as on the statement you're trying to prove. It's not an objective standard!

In this class, your actual audience is me (usually), but you should write proofs which are intended to convince your classmates. Write each proof as if you will have to present it at the board in front of the class (as, indeed, you occasionally will).

Now, typically statements one wants to prove are of the form $\forall x \in S,\ P(x) \Rightarrow Q(x)$. For example:

"If x is an odd integer, then $9x+5$ is even"

"Let $n \in \mathbb{N}$. If $15n$ is even, then $9n$ is even."

"All even natural numbers greater than 2 are the sum of 2 primes" (Goldbach Conjecture)

"Suppose $x, y, z, n \in \mathbb{N}$ such that $x^n + y^n = z^n$. Then $n \leq 2$." (Fermat's Last Theorem)

So let's see how this works.

Prop: For $n \in \mathbb{N}$, if $n + \frac{1}{n} < 2$, then $n^2 + \frac{1}{n^2} < 4$.

Proof: Suppose $n \in \mathbb{N}$. Then either $n = 1$ or $n \geq 2$. If $n = 1$ then $n + \frac{1}{n} = 1 + \frac{1}{1} = 2 \not< 2$. If $n \geq 2$ then $n + \frac{1}{n} > n \geq 2$.

Either way, $n + \frac{1}{n}$ is never < 2, so the statement is vacuously true. ■

Here, I'm using the fact that a conditional is automatically true if the antecedent is false

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Similarly, a result will be trivially true if the consequent is always true (remember $P \Rightarrow Q \equiv (\sim P) \vee Q$)

Prop: If n is an even integer, then $8n-6$ is even.

Proof: Suppose $n \in \mathbb{Z}$. Then $8n-6 = 2(4n-3)$ is a multiple of 2 & hence even, so the statement is trivially true. ▨

Notes:
- I start each proof by assuming the quantified variable is an element of the appropriate universe. This ensures that writer & reader are on the same page & makes the universe clear. This may seem excessive, but a shocking number of incorrect proofs boil down to a mistake in understanding the universe.
- I end each proof w/ the symbol ▨. This is (a) useful to the reader & (b) satisfying to the writer, so it's a good habit to get into. Old school mathematicians use "Q.E.D." rather than ▨, but this is becoming increasingly rare.