Math 3200: Day 8

Tautologies & contradictions are particularly important because a very common way of proving theorems is to show that either the statement of the theorem is equivalent to a tautology or to show that the negation of the statement is equivalent to a contradiction ("proof by contradiction").

What do we mean by "equivalent"? Well, if R & S are two compound statements involving the same component statements. Then R & S are logically equivalent if they have the same truth values for all combinations of truth values of their components.

e.g. $P \Rightarrow Q$ & $(\sim P) \vee Q$ are equivalent, which we write as $P \Rightarrow Q \equiv (\sim P) \vee Q$

Likewise: $P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$

Equivalently, $(P \Rightarrow Q) \Longleftrightarrow ((\sim Q) \Rightarrow (\sim P))$ is a tautology!

Here are three statements: "All primes are odd", "2 is prime", "2 is even".

Now, clearly there is a relationship b/w these statements. Indeed, the truth of the second & third should imply the falsity of the first. But how can we translate these into symbols in such a way that the relationship is clear?

If we just let P: "all primes are odd" Q: "2 is prime" R: "2 is even", then there's no semantic connection & no reason to believe, say, that $Q \wedge R \Rightarrow P$.

So we need to do better by going beyond so-called sentential logic and embrace monadic predicate logic... in particular, quantifiers.

First of all, let $P(x)$: x is prime & $O(x)$: x is odd. Then $P(2)$ is the statement "2 is prime" & $(\sim O(2))$ is the statement "2 is even".

But what about "All primes are odd"?

We can see that $P(x) \Rightarrow O(x)$ is the open statement "If x is prime, then x is odd." So "All primes are odd" is the statement "For all $x \in \mathbb{N}$, $P(x) \Rightarrow O(x)$." "For all" is an example of the use of the universal quantifier, symbolized as $\forall$.

Then we see that $\forall x \in \mathbb{N},\ P(x) \Rightarrow O(x)$ is a valid symbolization of "All primes are odd".

Now, what is the negation of this statement?

In English it's relatively easy. The negation of "All primes are odd" is surely "Some prime is even", or "There is some prime which is even".

The statement $P(x) \wedge (\sim O(x))$ is "x is prime & x is even"... to say there is some x for which this is true, we need the existential quantifier $\exists$.

Specifically, $\exists x \in \mathbb{N},\ P(x) \wedge (\sim O(x))$ is the statement "There exists $x \in \mathbb{N}$ so that x is prime & x is even."

This is the negation of $\forall x \in \mathbb{N},\ P(x) \Rightarrow O(x)$.

But what's the general pattern?

$$\sim\big(\forall x \in \mathbb{N},\ P(x) \Rightarrow O(x)\big) \equiv \big(\exists x \in \mathbb{N}, \sim(P(x) \Rightarrow O(x))\big) \equiv \exists x \in \mathbb{N},\ P(x) \wedge (\sim O(x))$$

In general, if $R(x)$ is an open sentence, then $\sim\big(\forall x \in S, R(x)\big) \equiv \exists x \in S, (\sim R(x))$

This makes perfect sense in English: "It's not the case that for all x, $R(x)$" is the same as "There is some x so that not $R(x)$".

Similarly, for the negation of an existential statement: $\sim\big(\exists x \in S,\ V(x)\big) \equiv \forall x \in S, (\sim V(x))$.