Math 3200: Day 42

Notation: If $|A| = \alpha$, we say that $|2^A| = 2^\alpha$.

Ex: Since $|\mathbb{N}| = \aleph_0$, we say that $|2^{\mathbb{N}}| = 2^{\aleph_0}$. Since $|\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}|$, this means $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

Now, since $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, we know that $\aleph_0 < 2^{\aleph_0}$. Also, we know that $\aleph_0 < c$ since $|\mathbb{N}| < |\mathbb{R}|$.

The natural question is: what's the relationship b/w $2^{\aleph_0}$ & c?

Thm: $2^{\aleph_0} = c$.

Proof: Since $2^{\aleph_0} = |\mathcal{P}(\mathbb{N})|$, it suffices to show that $|\mathcal{P}(\mathbb{N})| = |(0,1)| = |\mathbb{R}|$. To do that, we just need to show that

① $|\mathcal{P}(\mathbb{N})| \le |(0,1)|$ & ② $|(0,1)| \le |\mathcal{P}(\mathbb{N})|$

① To show $|\mathcal{P}(\mathbb{N})| \le |(0,1)|$, we need to give an injective map $f : \mathcal{P}(\mathbb{N}) \to (0,1)$. Define

$$f(A) = 0.a_1a_2a_3\dots$$

where $a_n = \begin{cases} 3 & \text{if } n \in A \\ 4 & \text{if } n \notin A \end{cases}$

This is injective: if $f(A) = f(B)$, then $0.a_1a_2\dots = 0.b_1b_2\dots$; since none of the digits can be 9, this means $a_n = b_n$ $\forall n \in \mathbb{N}$.

But that means $a_n \in A$ & $b_n \in B$ or $a_n \notin A$ & $b_n \notin B$ $\forall n \in \mathbb{N}$, so $A = B$ & f is injective.

Therefore, $|\mathcal{P}(\mathbb{N})| \le |(0,1)|$.

② To show $|(0,1)| \le |\mathcal{P}(\mathbb{N})|$, we need an injective map $g : (0,1) \to \mathcal{P}(\mathbb{N})$. Define

$$g(0.x_1x_2x_3,\dots\} = \{x_n 10^{n-1} : n \in \mathbb{N}\}.$$

(Recall that we're choosing the unique decimal representation of each elt of $(0,1)$ that does not end in an infinite list of 9s)

If $g(0.x_1x_2\dots) = g(0.y_1y_2\dots\}$, then $\{x_1, 10x_2, 100x_3, \dots\} = \{y_1, 10y_2, 100y_3, \dots\}$.

For each n, $10^{n-1}x_n$ is the only element of the first set in the interval $[10^{n-1}, 10^n)$; likewise, $10^{n-1}y_n$ is the only elt of the second set in this interval. Since the sets are equal, they contain the same elts;

since there's no way for $10^{n-1}x_n$ to be equal to $10^{m-1}y_m$ unless $n=m$, this means $x_n=y_n$ $\forall\, n\in\mathbb{N}$, so

$0.x_1x_2\ldots = 0.y_1y_2\ldots$ & hence g is injective. Therefore, $|(0,1)|\leq|\mathcal{P}(\mathbb{N})|$

Having proved $|\mathcal{P}(\mathbb{N})|\leq|(0,1)|$ & $|(0,1)|\leq|\mathcal{P}(\mathbb{N})|$, we have that $|\mathcal{P}(\mathbb{N})|=|(0,1)|$.

Since $|\mathcal{P}(\mathbb{N})|=2^{\aleph_0}$ & $|(0,1)|=|\mathbb{R}|=c$, we conclude that $2^{\aleph_0}=c$. $\boxtimes$

Let $\aleph_1$ be the next infinite cardinality after $\aleph_0$. In other words (by definition) there does not exist a set A so that

$$\aleph_0<|A|<\aleph_1.$$

Q: Is $\aleph_1=2^{\aleph_0}$?

This is a famous conjecture in mathematics, called the Continuum Hypothesis:

Continuum Hypothesis: $2^{\aleph_0}=\aleph_1$.

In 1940, Gödel showed that the Continuum Hypothesis cannot be disproved using the standard axioms of set theory (ZFC).

More surprisingly, in 1963 Paul Cohen showed that the Continuum Hypothesis cannot be proved using the standard axioms, either.