Math 3200: Day 41

We showed last time that $|A| < |\mathcal{P}(A)|$ for any set A. Now, what is $|\mathcal{P}(A)|$?

Well, if A is finite, we know that $|\mathcal{P}(A)| = 2^{|A|}$.

e.g., if $A = \{1,2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

How did we prove that $|\mathcal{P}(A)| = 2^{|A|}$ for finite sets?

One way to do it is to notice that we can count subsets by checking each element to see whether it's in the set or not:

if $S \subseteq A$ & $a \in A$, then either $a \in S$ or $a \notin S$. So there are 2 possibilities for each element of A; since there are $|A|$ elements of A, that means $2^{|A|}$ total possibilities.

Here's a slightly different way of saying the same thing: for each $S \subseteq A$, we can define a function $f_S : A \to \{0,1\}$ by:

$$f_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases}$$

Of course, we can do this regardless of whether or not A is finite.

In other words, this defines a map

$$F : \mathcal{P}(A) \longrightarrow \{0,1\}^A$$
$$S \longmapsto f_S$$

Def: We use the notation 2^A for the set $\{0,1\}^A$.

Ex: Let $A = \{a,b,c\}$. Then $F(\{a\}) = f_{\{a\}} = \{(a,1),(b,0),(c,0)\}$

$$F(\{b,c\}) = f_{\{b,c\}} = \{(a,0),(b,1),(c,1)\}$$

$$F(\emptyset) = f_\emptyset = \{(a,0),(b,0),(c,0)\} \qquad \text{etc.}$$

Notice that in this case $|2^A| = 8 = 2^{|A|}$.

Thm: For any nonempty set A, $|\mathcal{P}(A)| = |2^A|$.

Notice that $|\mathcal{P}(\emptyset)| = 1 = 2^0$, even though $2^\emptyset = \{0,1\}^\emptyset$ doesn't really mean anything.

Proof of Thm: As usual, we just need to find a bijective map from $\mathcal{P}(A)$ to 2^A. But of course this is just going to be our function F, which we just need to show is bijective.

Injectivity: Suppose $S, S' \subseteq A$ s.t. $F(S) = F(S')$. This means that $f_S(a) = f_{S'}(a)$ $\forall a \in A$. But then that means either $a \in S$ & $a \in S'$ or $a \notin S$ & $a \notin S'$ $\forall a \in A$, so $S = S'$ & F is injective.

Surjectivity: Let $\varphi \in 2^A$. In other words φ is a function $\varphi : A \to \{0,1\}$. Now define $B \subseteq A$ by

$$B = \{a \in A : \varphi(a) = 1\}$$

Then for each $a \in A$, $f_B(a) = \varphi(a)$, so $F(B) = f_B = \varphi$ & F is surjective.

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Since $F : \mathcal{P}(A) \to 2^A$ is bijective, we conclude that $|\mathcal{P}(A)| = |2^A|$.

We now know that $|\mathbb{N}| = \aleph_0$, $|\mathbb{R}| = c$ & $|\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| = 2^{\aleph_0}$.

Also, we know that $\aleph_0 < 2^{\aleph_0}$ & that $\aleph_0 < c$. So the natural question is: does $2^{\aleph_0} = c$?

Yes!

Thm: $2^{\aleph_0} = c$.