Math 3200: Day 36

Prop: $\mathbb{Z}$ is countable.

Proof: We need to find a bijective map $f: \mathbb{N} \to \mathbb{Z}$. For example, let f be determined by the arrows below:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

This map is obviously bijective, so $|\mathbb{Z}| = |\mathbb{N}|$. ∎

Here's a formula for the map $f: \mathbb{N} \to \mathbb{Z}$ from the above proof:

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}$$

Here's a cool theorem:

Thm: $\mathbb{Q}$ is countable.

Proof: For each $n \in \mathbb{N}$, define

$$A_n = \left\{ \pm\frac{p}{q} : p \in \mathbb{Z}_{\geq 0},\ q \in \mathbb{N},\ \frac{p}{q} \text{ in lowest terms},\ p+q = n \right\}$$

For example, $A_1 = \left\{\frac{0}{1}\right\}$, $A_2 = \left\{\frac{1}{1}, -\frac{1}{1}\right\}$, $A_3 = \left\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}\right\}$, etc.

Notice that each A_n is finite, since there are only finitely many ways to write n as the sum of 2 positive integers.

Now, just line them up back-to-back & define the map from $\mathbb{N}$ to $\mathbb{Q}$:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \;\cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \underbrace{\tfrac{0}{1}}_{A_1} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{1} \quad -\tfrac{1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{2}{1} \quad -\tfrac{2}{1}}_{A_3}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{3} \quad -\tfrac{1}{3} \quad \tfrac{3}{1} \quad -\tfrac{3}{1}}_{A_4}} & \tfrac{1}{4} \;\cdots \end{array}$$

This map is surjective since each element of $\mathbb{Q}$ appears in some A_n & it is injective since each element of $\mathbb{Q}$ appears in only one A_n. ∎

Alternatively, one can list the elements of $\mathbb{Q}^+$ in a 2-dim'l array:

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & \\
\hline
1 & 1/1 & 1/2 & 1/3 & 1/4 & \cdots \\
2 & 2/1 & 2/2 & 2/3 & 2/4 & \cdots \\
3 & 3/1 & 3/2 & 3/3 & 3/4 & \cdots \\
4 & 4/1 & 4/2 & 4/3 & 4/4 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

The red path traces out a surjective map $\mathbb{N} \to \mathbb{Q}^+$.

Lemma: If A is infinite & $\exists$ surjective $f: \mathbb{N} \to A$, then A is countable.

Then the Lemma implies $\mathbb{Q}^+$ is countable.

In this proof it's important not to traverse the rows... after all, you'd never get to the second row if you tried to get all of the first row first.

Here's an even more amazing theorem:

Thm (Cantor): $\mathbb{R}$ is uncountable.

This means there is no bijective map $\mathbb{N} \to \mathbb{R}$, so the cardinality of $\mathbb{R}$ is bigger than the cardinality of $\mathbb{N}$ (& hence, of $\mathbb{Q}$).