Math 3200: Day 35

Prop: If $f:A\to B$ & $g:B\to C$ are both injective, then $g\circ f:A\to C$ is injective.

Proof: Let $x,y\in A$ & suppose $(g\circ f)(x)=(g\circ f)(y)$. Then $g(f(x))=g(f(y))$ and so, since g is injective, we know that $f(x)=f(y)$.

But then, since f is injective, $x=y$ & we're done. ▨

Q: Suppose $f:A\to B$ & $g:B\to C$. If g is bijective, does it follow that $g\circ f$ is bijective?

A: No! Suppose $A=\{1,2\}$, $B=\{1,2,3\}$, $C=\{4,5,6\}$ & let $f:A\to B$ & $g:B\to C$

$$f:\ 1\mapsto 1,\quad 2\mapsto 2 \qquad\qquad g:\ 1\mapsto 4,\quad 2\mapsto 5,\quad 3\mapsto 6.$$

Then g is bijective, but $g\circ f$ is not bijective since it's not surjective: there is no $a\in A$ s.t. $(g\circ f)(a)=6$.

Though note that $g\circ f$ is injective, as guaranteed by the proposition.

Cardinality

As suggested, we will generalize the notion of cardinality by saying that two sets have the same cardinality if there exists a bijective map between them.

Prop: Let $\mathcal{S}$ be a nonempty collection of nonempty sets. Define the relation R on $\mathcal{S}$ by ARB if there exists a bijective map from A to B.

Then R is an equivalence relation.

Proof: Let $A\in\mathcal{S}$. Then the identity map $i_A:A\to A$ given by $i_A(a)=a$ is bijective, so ARA & R is reflexive.

Let $A,B\in\mathcal{S}$ so that ARB. Then $\exists$ bijective $f:A\to B$. But then $f^{-1}:B\to A$ is bijective, so BRA & R is symmetric.

Let $A,B,C\in\mathcal{S}$ so that ARB & BRC. Then $\exists$ bijective $f:A\to B$ & $g:B\to C$. But then $g\circ f:A\to C$ is bijective

(by HW problem 9.42(a)), so ARC & R is transitive. ▨

Remark: In the above theorem we couldn't let $\mathcal{S}$ be the set of all sets because such a thing can't exist:

Russell's Paradox shows that the collection of all sets is not itself a set (instead it is what's called a proper class).

We say that $|A|=|B|$ if $\exists$ bijective map from A to B.

Def: A set A is **countable** if $|A| = |\mathbb{N}|$ (the book calls this **denumerable**).

Prop: $\mathbb{Z}$ is countable.