Math 3200: Day 33

Prop: Let A & B be finite sets. There exists an injective map $f: A \to B$ if & only if $|A| \leq |B|$.

Proof: ($\Rightarrow$) Suppose there exists injective $f: A \to B$. Since $f(A) \subseteq B$, we know $|f(A)| \leq |B|$. But since f is injective,

$$|A| = |f(A)| \leq |B|.$$

($\Leftarrow$) On the other hand, suppose $|A| \leq |B|$. Then $A = \{a_1, \ldots, a_k\}$ & $B = \{b_1, \ldots, b_n\}$ for $k \leq n$. Then define

$f: A \to B$ by $f(a_i) = b_i$. If $f(a_i) = f(a_j)$, then $b_i = b_j$ so $i = j$; hence, f is injective. ∎

Prop: Let A & B be finite sets. There exists a surjective map $g: A \to B$ if & only if $|A| \geq |B|$.

Proof: ($\Rightarrow$) Suppose $g: A \to B$ is surjective. Then $g(A) = B$, so $|A| \geq |g(A)| = |B|$.

($\Leftarrow$) Suppose $|A| \geq |B|$. Then $A = \{a_1, \ldots, a_k\}$ & $B = \{b_1, \ldots, b_n\}$ for $k \geq n$. Define

$g: A \to B$ by $g(a_i) = \begin{cases} b_i & \text{if } i \leq n \\ b_n & \text{if } i > n \end{cases}$ Then g is obviously surjective. ∎

Def: A map $f: A \to B$ that is both injective & surjective is called **bijective**.

An immediate corollary of the propositions above:

Corollary: Let A & B be finite sets. There exists a bijective map $h: A \to B$ if & only if $|A| = |B|$.

Note: This is the notion of cardinality which generalizes to infinite sets.