Math 3200: Day 31

Functions:

Q: What is a function?

A: ... fundamentally, a function is a rule that assigns each element of some set A an element of a set B.

One way of saying this is that a function $f: A \to B$ is a very special type of relation from A to B (note: this is in some sense at the opposite end of the relation spectrum from an equivalence relation, and this will almost never be equivalence relations).

Namely, if we regard f as a subset of $A \times B$, then $\forall a \in A$, $\exists!\ b \in B$ s.t. $(a,b) \in f$.

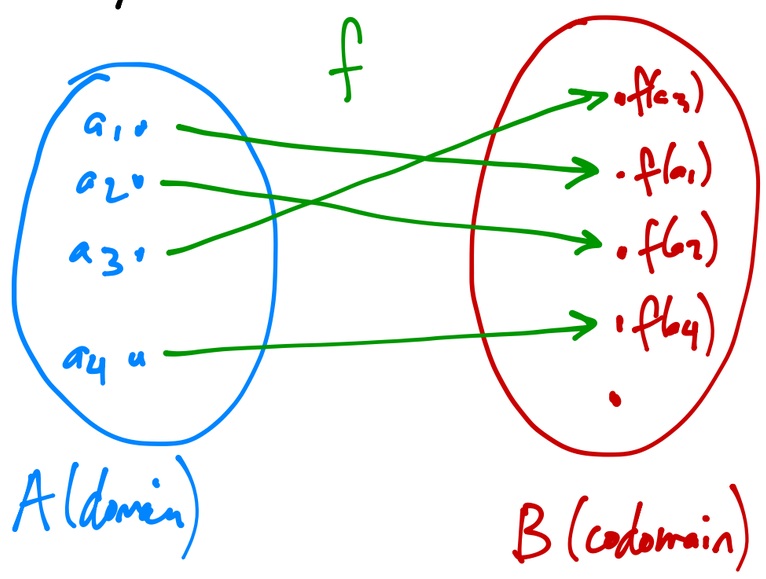
Let's unpack this a little bit. This is saying that ① we can stick **every** elmt of A (called the **domain** of f) into f.

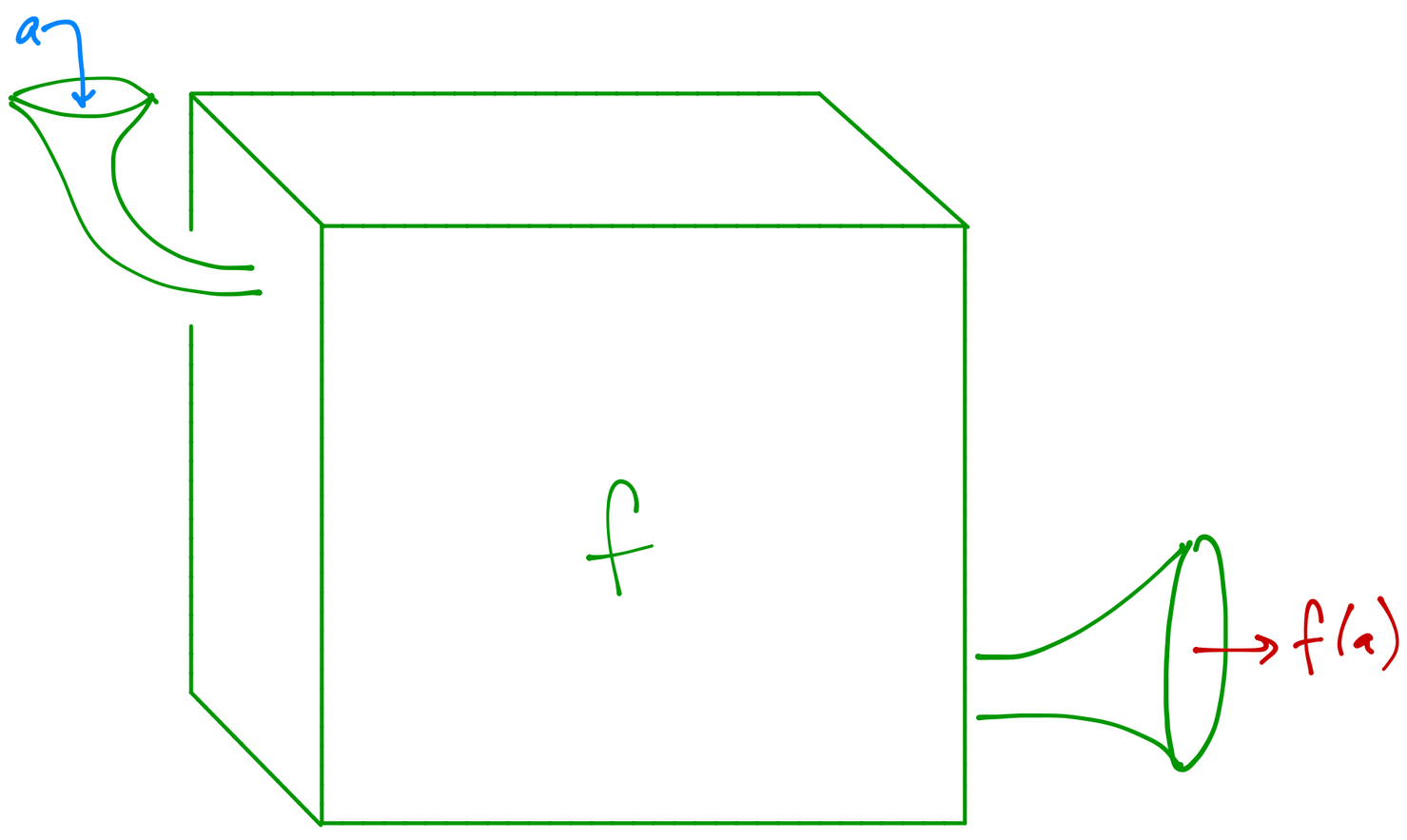
② For any given $a \in A$, there is **only one** ordered pair of the form (a,b) in f.

In other words, the rule for what to send a is deterministic.

Notationally, if $(a,b) \in f$, we say that $f(a) = b$.

Two visual metaphors:





Ex: • $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 1$. i.e., $f = \{(x, x^2-1) : x \in \mathbb{R}\}$

Notice that $\mathbb{R}$ is both domain & codomain, but the **range** of f is $\{y \in \mathbb{R} : y \geq -1\}$

• $g_n: \mathbb{Z} \to \mathbb{C}$, $k \mapsto e^{\frac{2\pi i k}{n}}$ Here, the range of g_n is the n^{th} roots of unity

- $r : \mathbb{R}^3 \to \mathbb{R}_+$

  $(x,y,z) \mapsto \sqrt{x^2+y^2+z^2}$

- $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

  $\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

- $h : \mathbb{C} \to [0,2\pi)$

  $re^{i\theta} \mapsto \theta$

- $D : [0,1] \to \{0,1\}$ by $D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ (the Dirichlet function)

Def: If $f : A \to B$ is a function & $C \subseteq B$, then the **inverse image** of C is the set

$$f^{-1}(C) = \{a \in A : f(a) \in C\} \subseteq A$$

Ex: · For $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, the inverse image of $(1,2)$ is the set $f^{-1}((1,2)) = (-1,-\sqrt{2}) \cup (1,\sqrt{2})$

the inverse image of $\{0,1,2,3\}$ is $f^{-1}(\{0,1,2,3\}) = \{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}\}$

· For $r : \mathbb{R}^3 \to \mathbb{R}_+$ as above, $r^{-1}(\{1\}) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ is the unit sphere.