Math 3200: Day 3

Taking unions or intersections of 2 sets is straightforward, but things can get messy if we have lots of sets.

For example, suppose $A_n = [n, n+1]$ for all $n \in \mathbb{Z}$. Then it's a pain to write, e.g.,

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7$, which of course equals...? $\left(\{x : x \in A_1 \text{ or } x \in A_2 \text{ or} \dots \text{or } x \in A_7\} = [1, 8] \right)$

Instead, we introduce the notation

$$\bigcup_{n=1}^{7} A_n = \{x : x \in A_n \text{ for some } n \text{ s.t. } 1 \leq n \leq 7\}$$

Even better, we can take infinite unions:

$$\bigcup_{n=1}^{\infty} A_n = ? \quad \left([1, +\infty) \right)$$

$$\bigcup_{n=-12}^{\infty} A_n = ? \quad \left([-12, \infty) \right)$$

$$\bigcup_{n=-\infty}^{5} A_n = ? \quad \left((-\infty, 5] \right)$$

$$\bigcup_{n=-\infty}^{\infty} A_n = ? \quad \left(\mathbb{R} \right)$$

Even more generally, for any index set I and indexed collection of sets $\{A_\alpha\}_{\alpha \in I}$, we can define the indexed union

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

Similarly for intersections:

$$\bigcap_{\alpha \in I} A_\alpha = ? \quad (\text{Discussion...})$$

Def: $$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for every } \alpha \in I\}$$

Q: Non-trivial example?

A: e.g., for each $n \in \mathbb{N}$, let $B_n = [0, 1/n]$. Then

$$\bigcap_{n\in\mathbb{N}} B_n = \{x : x \in B_n \text{ for any } n\} = \{x : x \in [0,1/n] \text{ for all } n\},$$ which pretty clearly equals the singleton set $\{0\}$.

With $A_n = [n, n+1]$ as before, notice that

$$\bigcap_{n\in\mathbb{Z}} A_n = \bigcap_{n\in\mathbb{Z}} [n, n+1] = \{x : x \in [n,n+1] \text{ for all } n\} = \emptyset.$$

Def: A partition of a set A is a collection of sets $\{A_\alpha : \alpha \in I\}$ for some indexing set I so that:

① $A_\alpha \subseteq A$ for all $\alpha \in I$

② $A = \bigcup_{\alpha\in I} A_\alpha$

③ $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha, \beta \in I$.

Q: Example of a partition? of an infinite set?

(e.g. $\{1,2,3\} = \{1,2\} \cup \{3\}$, $\mathbb{N} = \{2n : n \in \mathbb{N}\} \cup \{2n-1 : n \in \mathbb{N}\}$, etc.)

Note: The order in which you list the elements of a set doesn't matter. So, for example, $\{1,2,3\} = \{3,2,1\} = \{3,1,2\} = \ldots$

However, in many cases the order does matter: to specify a point (x,y) in the plane, the order is very important.

The point $(1,5)$ is not the same as the point $(5,1)$!

In higher dimensions, of course, we might have arbitrarily many coordinates $(x_1, \ldots, x_n)$.

To be able to talk about such ordered pairs (or, in general, ordered n-tuples), we need Cartesian Products:

Def: The Cartesian product $A \times B$ of sets A & B is the set of all ordered pairs w/ the first term coming from A & the second term coming from B. In other words,

$$A \times B = \{(a,b) : a \in A \text{ \& } b \in B\}.$$

Ex: · Let $A = \{1, \{1\}\}$, $B = \{2\}$. Then

$$A \times B = \{(a,b) : a \in A, b \in B\} = \{(1,2), (\{1\},2)\}.$$

· $\mathbb{R} \times \mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}$ precisely describes the plane!

· $\mathbb{Z} \times \mathbb{Z} = \{(a,b) : a,b \in \mathbb{Z}\}$ is the integer lattice. Notice that $\mathbb{Z} \times \mathbb{Z} \subseteq \mathbb{R} \times \mathbb{R}$.