Math 3200: Day 22

Prop: For all integers $n \geq 0$, $3 \mid (2^n + 2^{n+1})$.

Proof: By induction. Let $P(n)$ be the statement that $3 \mid (2^n + 2^{n+1})$. Then the goal is to prove $P(n)$ is true for all integers $n \geq 0$.

Base Case: Since this is a statement about nonnegative integers, the base case is $P(0)$, which says that $3 \mid (2^0 + 2^{0+1})$, which is obv. true.

Inductive Step: Let $k \in \{0, 1, 2, \ldots\}$ and assume $P(k)$ is true, meaning that $3 \mid (2^k + 2^{k+1})$. In other words, $2^k + 2^{k+1} = 3l$ for some $l \in \mathbb{Z}$.

Then $2^{k+1} + 2^{(k+1)+1} = 2(2^k + 2^{k+1}) = 2(3l) = 3(2l)$, so $2^{k+1} + 2^{(k+1)+1}$ is a multiple of 3, which means $P(k+1)$ is true.

Having proved both the base case & the inductive step, the principle of mathematical induction allows us to conclude that $3 \mid (2^n + 2^{n+1})$ $\forall\, n \geq 0$. ▨

Sometimes we need a slightly stronger version of induction:

Principle of Strong Induction: For each $n \in \mathbb{N}$, let $P(n)$ be a statement. If

(i) $P(1)$ is true and

(ii) For each $k \in \mathbb{N}$, if $P(i)$ is true for all $1 \leq i \leq k$, then $P(k+1)$ is true

then $P(n)$ is true $\forall\, n \in \mathbb{N}$.

Example: Consider the Fibonacci sequence $F_1, F_2, \ldots$ defined recursively by

$$F_1 = 1,\ F_2 = 1,\quad F_n + F_{n+1} = F_{n+2}\ \forall\, n \in \mathbb{N}.$$

So the sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

This sequence arises in a surprising number of contexts...

Prop: For all $n \in \mathbb{N}$, the n^{th} Fibonacci number is given by

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Note: This is an utterly amazing fact. It means we can figure out, say, the millionth Fibonacci # w/o having to know the first $999{,}999$, which is obviously a big time-saver. (If you could figure out a similar formula for the n^{th} prime number, you would instantly become the most famous mathematician in the world.)

Proof of Prop: By strong induction. Let $P(n)$ be the statement that $F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$.

Base Case: $P(1)$ says that $F_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$, which is clearly true since both sides equal 1.

Inductive Step: Let $k \in \mathbb{N}$ & suppose $P(i)$ is true for all $1 \leq i \leq k$.

We want to prove $P(k+1)$, so consider

$$F_{k+1} = F_k + F_{k-1}.$$

For notational ease, let $\varphi = \frac{1+\sqrt{5}}{2}$ & $\psi = \frac{1-\sqrt{5}}{2}$. Then the inductive hyp. implies

$$F_k = \frac{\varphi^k - \psi^k}{\sqrt{5}} \quad \& \quad F_{k-1} = \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}}.$$

So $$F_{k+1} = F_k + F_{k-1} = \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} = \frac{(\varphi^k + \varphi^{k-1}) - (\psi^k + \psi^{k-1})}{\sqrt{5}} = \frac{\varphi^{k-1}(\varphi+1) - \psi^{k-1}(\psi+1)}{\sqrt{5}}.$$

Clearly, we would be done if $\varphi + 1 = \varphi^2$ & $\psi + 1 = \psi^2$. This turns out to be true, as we'll prove in the below lemma.

Therefore, strong induction allows us to conclude that $P(n)$ is true $\forall\, n \in \mathbb{N}$. ∎

Lemma: $\varphi + 1 = \varphi^2$ & $\psi + 1 = \psi^2$.

Proof: Clearly, $\varphi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$. OTOH,

$$\varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}.$$

Similarly for ψ. ∎