Math 3200: Day 21

Problems to present: 5.14, 5.26

What went wrong w/ the proof I presented last time that all positive integers are equal?

A: The proof of the inductive step contained the following:

Let $x, y \in \mathbb{N}$ so that $\max(x,y) = k+1$. Then $\max(x-1, y-1) = \max(x,y) - 1 = (k+1) - 1 = k$. Since we assumed $P(k)$ is true, $x-1 = y-1 \Rightarrow x = y$.

Here's the problem: $P(k)$ says if $x, y \in \mathbb{N}$ s.t. $\max(x,y) = k$, then $x = y$. But in the above if, say, $x = 1$, then $x - 1 \notin \mathbb{N}$, so $P(k)$ says nothing about $x-1$ & $y-1$.

Things can also go very wrong if you leave out the base case:

Claim: All natural numbers are both even & odd.

"Proof": If $k \in \mathbb{N}$ is even, then $k+1$ is odd; also, if k is odd, then $k+1$ is even.

Therefore, whenever k is both even & odd, so is $k+1$, and then $k+2$, and so on. ☒

Note: This is a completely valid proof of the inductive step... but of course there's no way to ever get a base case.

Why is induction true?

Well-Ordering Principle: Every subset of $\mathbb{N}$ has a smallest element.

Again, this is an axiom, not a theorem, so there's no proof of the above fact... it's just obviously true (or, if you like, one of the existing rules we build into the definition of $\mathbb{N}$).

Principle of Mathematical Induction: For each $n \in \mathbb{N}$, let $P(n)$ be a statement. If

(i) $P(1)$ is true and

(ii) $P(k) \Rightarrow P(k+1)$ is true for each $k \in \mathbb{N}$

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose, for the sake of contradiction, that $P(1)$ is true & $P(k) \Rightarrow P(k+1)$ is true $\forall k \in \mathbb{N}$, but $P(n)$ is <u>not</u> true for all $n \in \mathbb{N}$.

Let $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. Then $S \neq \emptyset$ & so, by the Well-Ordering Principle, S contains a least element m.
(This is called the minimum counterexample).

Since $m \in S$, $P(m)$ is false. But since m is the smallest element of S, $m-1 \notin S$, so $P(m-1)$ is true.

But then $P(m-1) \Rightarrow P(m)$ $(\Rightarrow\Leftarrow)$

From this contradiction, we can conclude that the negation of the PMI is false, so PMI is true. $\blacksquare$