Math 3200: Day 20

The Principle of Mathematical Induction: For each positive integer n let $P(n)$ be a statement. If

(i) $P(1)$ is true and

(ii) $P(k) \Rightarrow P(k+1)$ is true for all $k \in \mathbb{N}$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

This is incredibly useful. Let's see a couple of examples before we see why the principle is true.

Prop: For any $n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

The claim is that we can prove this by induction, so we should have some P's floating around here.

In this case, we want the conclusion to be that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$, so we should let $P(n)$ be the statement

"$\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$".

So then the 2 parts of the proof are to show that $P(1)$ is true & that $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$.

Proof: Let $P(n)$ be the statement that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. I will prove that $P(n)$ is true for all $n \in \mathbb{N}$ using induction.

Base Case: $P(1)$ says that $\displaystyle\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$; i.e., that $1 = \frac{2}{2}$, which is clearly true.

Inductive Step: Let $k \in \mathbb{N}$ and suppose that $P(k)$ is true. In other words, suppose that $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

The goal is to use this information to show that $P(k+1)$ is true; i.e., that $\displaystyle\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$.

So consider

$$\sum_{i=1}^{k+1} i = 1+2+\cdots+k+(k+1) = (1+2+\ldots+k)+(k+1) = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1),$$

where the substitution highlighted in green is exactly the inductive assumption.

In other words,

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2k+2}{2} = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2},$$

so we see that $P(k+1)$ is true.

Therefore, having proved the base case & the inductive step, the Principle of Mathematical Induction allows me to conclude that $P(n)$ is true for all $n \in \mathbb{N}$, which is exactly the statement of the proposition. $\blacksquare$

A fake proof by induction

Claim: All positive integers are equal.

"Proof": Let $P(n)$ be the statement

"For any $x, y \in \mathbb{N}$, if $\max(x,y) = n$, then $x = y$."

Base Case: $P(1)$ says "For $x, y \in \mathbb{N}$, if $\max(x,y) = 1$, then $x = y$." This is clearly true, since $\max(x,y) = 1 \Leftrightarrow x = y = 1$.

Inductive Step: Let $k \in \mathbb{N}$ & assume $P(k)$ is true.

Let $x, y \in \mathbb{N}$ so that $\max(x,y) = k+1$. Then $\max(x-1, y-1) = \max(x,y) - 1 = (k+1) - 1 = k$. Since we assumed $P(k)$ is true, $x-1 = y-1 \Rightarrow x = y$.

By the Principle of Math. Ind., $P(n)$ is true for all $n \in \mathbb{N}$. But then, since $\max(1,n) = n$ $\forall n \in \mathbb{N}$, we see that $n = 1$ $\forall n \in \mathbb{N}$. $\blacksquare$

So what's the problem w/ this?