Math 3200: Day 19

T/F: There exists a real solution to the equation $x^4-2x^2-1=0$.

Usually a good idea to do some simple examples... if you get lucky, one of the examples will actually be a solution, but if not the examples should give some insight.

If $x=1$, then $x^4-2x^2-1=1-2-1=-2$

$x=-1$, $x^4-2x^2-1=1-2-1=-2$

$x=0$, $x^4-2x^2-1=-1$

$x=\sqrt{2}$, $x^4-2x^2-1=4-4-1=-1$

$x=2$, $x^4-2x^2-1=16-8-1=7$

Aha! Now we still don't know what a solution should be exactly, but we should expect that there is a solution b/w $x=\sqrt{2}$ & $x=2$ since x^2-2x-1 changes sign on this interval. Recall:

Intermediate Value Thm: If $f(x)$ is a continuous function on the interval $[a,b]$, then for every number M b/w $f(a)$ & $f(b)$, there exists $c\in(a,b)$ so that $f(c)=M$.

In this case, if $f(x)=x^4-2x^2-1$, then 0 is b/w $f(\sqrt{2})=-1$ & $f(2)=7$. Since polynomials are always continuous, there must be some $c\in(\sqrt{2},2)$ so that $f(c)=0$.

So the above statement is True, and here's the proof:

Proof: Let $f(x)=x^4-2x^2-1$. Then $f(x)$ is a polynomial and hence continuous. Note that $f(\sqrt{2})=-1$ & that $f(2)=7$.

Since $-1<0<7$ and since f is continuous, the Intermediate Value Thm implies that there exists $c\in(\sqrt{2},2)$ so that $f(c)=0$. The number c is clearly real & hence is a real solution of the equation $x^4-2x^2-1=0$. $\blacksquare$

Notice that this proof is non-constructive: we've proved the existence of a real solution, but we haven't actually found one explicitly.

Well, can we do better? Here's a slightly harder question:

T/F: There is a rational solution of $x^4-2x^2-1=0$.

We've already done the simplest examples & didn't find a rational solution, so it's probably a sucker's bet to keep plugging in rational numbers.

Instead, let's assume there is a rational solution p/q in lowest terms and see where that assumption leads us. If it leads to conditions on p & q then great, we'll know where to look; if it leads to a contradiction, then that will be a proof by contradiction that there is no rational solution.

Suppose p/q is a rational sol'n in lowest terms of $x^4-2x^2-1=0$. Then

$$0=\left(\frac{p}{q}\right)^4-2\left(\frac{p}{q}\right)^2-1=\frac{p^4-2p^2q^2-q^4}{q^4}.$$

so $0=p^4-2p^2q^2-q^4$.

Now, this is almost $(p^2-q^2)^2$, so add & subtract q^4 to get

$$0=p^4-2p^2q^2+q^4-2q^4=(p^2-q^2)^2-2q^4.$$

In other words, $(p^2-q^2)^2=2q^4$ is even. Hence, p^2-q^2 is even, so $p^2-q^2=2n$ for some $n\in\mathbb{Z}$, meaning that

$2q^4=(p^2-q^2)^2=(2n)^2=4n^2$ & $q^4=2n^2$ is even. In turn, this means q is even.

On the other hand, p^2-q^2 being even means, by Problem 3.30 from previous HW, that p^2 & q^2 have the same parity & hence that p & q have the same parity. Since we just saw that q is even, this means p & q are both even. But then this contradicts the fact that p/q is in lowest terms.

From this contradiction, then, we see that $x^4-2x^2-1=0$ has no rational solutions

Add the word "Proof" before "Suppose p/q is a rational solution..." and a "$\blacksquare$" at the end of the preceding sentence, & we've just produced a proof by contradiction that there is no rational solution of $x^4+2x^2+1=0$.