Math 3200: Day 16

Exam 1 grade breakdown:

A	>24
B	21-24
C	18-21
D/F	<18

So far, most of our proofs have been about numbers, but of course numbers aren't the only things we can make & prove statements about. Today we'll talk a bit about how to prove statements about sets.

Prop: For any sets A & B, $A \cap B = A - (A - B)$.

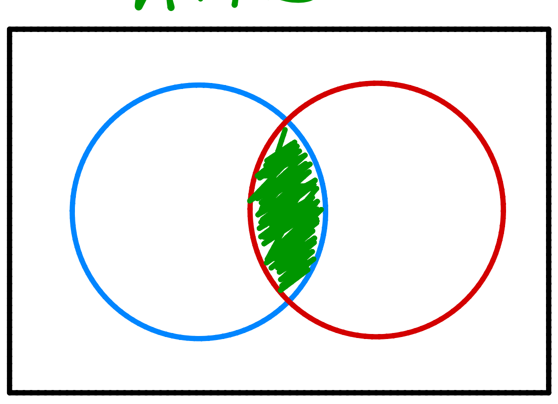
Before we try to prove this, first let's think about what it even means for 2 sets to be equal: if S & T are sets & we claim that $S = T$, then that means every element of S is an element of T and that every element of T is an element of S. So to prove that $S = T$, the standard strategy is a 2-part approach:

① Show that every $s \in S$ is an element of T, meaning that $S \subseteq T$

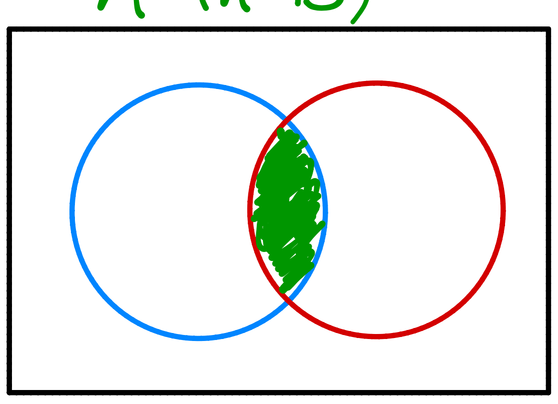
② Show that every $t \in T$ is an element of S, meaning that $T \subseteq S$.

Before we try to implement this strategy, it's probably a good idea to convince ourselves that the proposition is true. So, for example, we could draw the Venn diagrams:

$A \cap B$



$A-(A-B)$



Proof of Prop: First, suppose that $x \in A \cap B$. Then, since $x \in B$, we see that $x \notin A - B$. But then, since $x \in A$ we see that $x \in A-(A-B)$. Since our choice of x was arbitrary, the same applies to all elements of $A \cap B$, so $A \cap B \subseteq A-(A-B)$.

On the other hand, suppose $x \in A-(A-B)$. Then $x \in A$ and $x \notin A-B$. But this can only be true if x is also an element of B, so $x \in A \cap B$. Since x was arbitrary, this means that $A-(A-B) \subseteq A \cap B$. ∎

Having proved inclusion both ways, we conclude that $A \cap B = A-(A-B)$.

Notice that essentially the same proof would have told us that $A \cap B = B-(B-A)$.

Prop: For any sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Just to have an example in mind, suppose $A = \{1,2,3\}$, $B = \{3,4,5\}$, $C = \{4,5,6\}$. Then

$B \cap C = \{4,5\}$ and $A \times (B \cap C) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$.

OTOH, $A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\}$ and $A \times C = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$

so $(A \times B) \cap (A \times C) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$ as claimed.

The strategy for proving the proposition is the usual:

Proof of Prop: First, suppose $(a,d) \in A \times (B \cap C)$. Then by definition $d \in B \cap C$, so $d \in B$ & $d \in C$, so $(a,d) \in A \times B$ & $(a,d) \in A \times C$, so $(a,d) \in (A \times B) \cap (A \times C)$.

Since the choice of (a,d) was arbitrary, it follows that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

On the other hand, suppose $(a,d) \in (A \times B) \cap (A \times C)$. Then $a \in A$ & $d \in B$ & $d \in C$, so $d \in B \cap C$, so $(a,d) \in A \times (B \cap C)$

Therefore, since the choice of (a,d) was arbitrary, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Since we proved containment both directions, we've shown that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. ∎