# Math 3200: Day 15

(A proof that requires some cleverness)

**Prop (Arithmetic/Geometric Mean Inequality):** If $a, b \in \mathbb{R}$ are positive, then

$$0 < \sqrt{ab} \leq \frac{a+b}{2}.$$

**Q:** Ideas on how to start? (What are the key assumptions?)

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**Hint 1:** Since $a$ & $b$ are positive, there exist $r, s > 0$ so that $a = r^2$, $b = s^2$.

**Hint 2:** Try re-writing $\sqrt{ab} \leq \frac{a+b}{2}$ as $0 \leq a - 2\sqrt{ab} + b$ & then re-write in terms of $r$ & $s$.

**A:** The key is to note that, if $a = r^2$ & $b = s^2$, then $0 \leq (r-s)^2 = r^2 - 2rs + s^2 = a - 2\sqrt{a}\sqrt{b} + b = a - 2\sqrt{ab} + b$.

**Proof:** Let $a, b \in \mathbb{R}$ be positive. Since $a$ & $b$ are positive, $\sqrt{a}$ & $\sqrt{b}$ are positive, so

$$0 < \sqrt{a}\sqrt{b} = \sqrt{ab},$$

taking care of the first inequality. Moreover,

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{a}\sqrt{b} + b = a - 2\sqrt{ab} + b.$$

Adding $2\sqrt{ab}$ to both sides & dividing by $2$ yields

$$\sqrt{ab} \leq \frac{a+b}{2},$$

which is the second inequality. ▨

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So far, most of our proofs have been about numbers, but of course numbers aren't the only things we can make & prove statements about. Today we'll talk a bit about how to prove statements about sets.

**Prop:** For any sets $A$ & $B$, $A \cap B = A - (A - B)$.

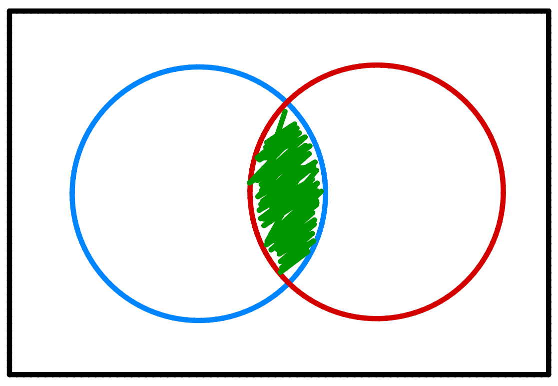
Before we try to prove this, first let's think about what it even means for 2 sets to be equal: if $S$ & $T$ are sets & we claim that $S = T$, then that means every element of $S$ is an element of $T$ and that every element of $T$ is an element of $S$. So to prove that $S = T$, the standard strategy is a 2-part approach:

① Show that every $s \in S$ is an elt of $T$, meaning that $S \subseteq T$

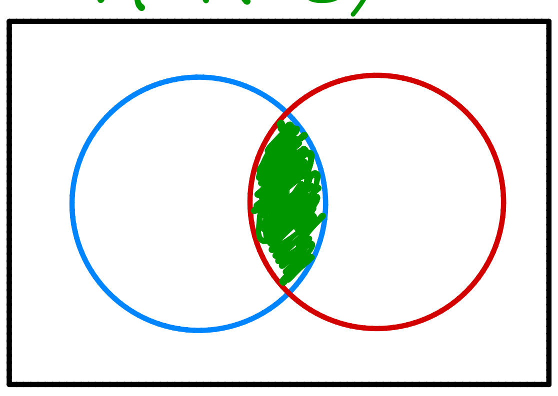
② Show that every $t \in T$ is an elt of $S$, meaning that $T \subseteq S$.

Before we try to implement this strategy, it's probably a good idea to convince ourselves that the proposition is true. So, for example, we could draw the Venn diagrams:

$A \cap B$



$A - (A - B)$



**Proof of Prop:** First, suppose that $x \in A \cap B$. Then, since $x \in B$, we see that $x \notin A - B$. But then, since $x \in A$ we see that $x \in A - (A - B)$. Since our choice of $x$ was arbitrary, the same applies to <u>all</u> elements of $A \cap B$, so $A \cap B \subseteq A - (A - B)$.

On the other hand, suppose $x \in A - (A - B)$. Then $x \in A$ and $x \notin A - B$. But this can only be true if $x$ is also an element of $B$, so $x \in A \cap B$. Since $x$ was arbitrary, this means that $A - (A - B) \subseteq A \cap B$. ∎

Having proved inclusion both ways, we conclude that $A \cap B = A - (A - B)$.

Notice that essentially the same proof would have told us that $A \cap B = B - (B - A)$.