# Math 3200: Day 11

Sometimes, the direct approach to proving something isn't the best. e.g.:

**Prop:** If $x, y \in \mathbb{Z}$ so that $xy$ is even, then at least one of $x$ or $y$ must be even.

This is something we all "know", but how can we actually prove it?

Note that $xy$ being even means $xy = 2a$ for some $a$. But how can we use this fact to conclude that either $x$ or $y$ is even?

One thing to notice is that the negation of "at least one of $x$ or $y$ must be even" is easier to work with.

In fact, if we assume that $x$ & $y$ are both odd, then it's straightforward to show that $xy$ is odd. But this is the contrapositive of the original statement!

(Recall that $P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$)

**Proof of Prop:** By contrapositive. Suppose $x$ & $y$ are both odd integers. Then $x = 2n+1$ & $y = 2m+1$ for some integers $n$ & $m$. But then

$$xy = (2n+1)(2m+1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$$

is odd. Therefore, we've shown that if $x$ & $y$ are odd then $xy$ is odd or, equivalently, if $xy$ is even, then at least one of $x$ or $y$ is even. $\blacksquare$

**Note:** The realization that it might be good to prove this by contrapositive came after some experimentation.

Very broadly speaking, if the (negation of the) consequent is easier to deal with you may be well-served to prove a statement by contrapositive, but there are no hard & fast rules.

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**Prop:** Suppose $x \in \mathbb{Z}$. Then $3x^2 - 1$ is even if & only if $x$ is odd.

**Q:** What do we need to do?

**A:** Since it's a biconditional, we need to prove 2 statements: ① If $3x^2-1$ is even, then $x$ is odd.

② If $x$ is odd, then $3x^2-1$ is even

The second is straightforward, but what's the strategy for proving the first?

(After some thinking about it...) A: Again, this is a natural for proving the contrapositive. It's much more straightforward to prove that if $x$ is even, then $3x^2-1$ is odd.

Proof of Prop: ($\Leftarrow$) Suppose $x \in \mathbb{Z}$ is odd. Then $x = 2n-1$ for some $n \in \mathbb{Z}$ and

$$3x^2-1 = 3(2n-1)^2-1 = 3(4n^2-4n+1)-1 = 12n^2-12n+3-1 = 12n^2-12n+2$$
$$= 2(6n^2-6n+1),$$

which is even. This proves that if $x$ is odd, then $3x^2-1$ is even.

($\Rightarrow$) Conversely, suppose $x \in \mathbb{Z}$ is even. Then $x = 2m$ for some $m \in \mathbb{Z}$ and

$$3x^2-1 = 3(2m)^2-1 = 12m^2-1 = 2(6m^2)-1,$$

which is odd. This proves that if $x$ is even, then $3x^2-1$ is odd or, equivalently, that if $3x^2-1$ is even, then $x$ is odd.

Having proved both directions of the biconditional, this completes the proof. $\blacksquare$