Math 3200 Exam #3 Practice Problem Solutions

1. Suppose $[a], [b] \in \mathbb{Z}_6$ so that $[a] \cdot [b] = [0]$. Can you conclude that either $[a] = [0]$ or $[b] = [0]$? Why or why not?

   Answer: No, you can't conclude that either $[a]$ or $[b]$ is $[0]$. For example, if $[a] = [2]$ and $[b] = [3]$, then

   $$[2] \cdot [3] = [2 \cdot 3] = [6] = [0].$$

2. List all the possible equivalence relations on the set $A = \{a, b\}$. (For organizational purposes, it may be helpful to write the relations as subsets of $A \times A$.)

   Answer: Thinking of an equivalence relation R on A as a subset of $A \times A$, the fact that R is reflexive means that

   $$\{(a,a),(b,b)\} \subseteq R.$$

   Clearly, one possibility is just to let $R = \{(a,a),(b,b)\}$, which is automatically reflexive, symmetric, and transitive.

   Now, if I want to add an element to R, my only possibilities are (a,b) and (b,a). But R being symmetric means that if I add one I have to add both, so I could also let $R = \{(a,a),(b,b),(a,b),(b,a)\}$. This is reflexive and symmetric by construction and it is automatically transitive since R only contains 2 elements.

   These are the only two possibilities, so I see that the only equivalence relations on A are

   $$\{(a,a),(b,b)\} \text{ and } \{(a,a),(b,b),(a,b),(b,a)\}.$$

3. Define the relation R on $\mathbb{Z}$ by $x \, R \, y$ if $x^2 \equiv y^2 \pmod 4$. Is R an equivalence relation? If so, what are the equivalence classes of R?

   Answer: Yes, R is an equivalence relation. To prove this, I need to show that R is reflexive, symmetric, and transitive.

   Reflexive: Let $x \in \mathbb{Z}$. Then $x^2 \equiv x^2 \pmod 4$, so $x \, R \, x$.

   Symmetric: Let $x, y \in \mathbb{Z}$ so that $x \, R \, y$. This means that $x^2 \equiv y^2 \pmod 4$, which obviously means that $y^2 \equiv x^2 \pmod 4$ and hence that $y \, R \, x$, so R is symmetric.

   Transitive: Let $x, y, z \in \mathbb{Z}$ so that $x \, R \, y$ and $y \, R \, z$. Then $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$, so we have that

   $$x^2 \equiv y^2 \equiv z^2 \pmod 4,$$

   so $x \, R \, z$ and we conclude that R is transitive.

   Now, to figure out the equivalence classes, let's think about the four possibilities for an integer: it can be congruent to 0, 1, 2, or 3 modulo 4.

   - If $a \equiv 0 \pmod 4$, then $a^2 \equiv 0^2 \equiv 0 \pmod 4$.
   - If $a \equiv 1 \pmod 4$, then $a^2 \equiv 1^2 \equiv 1 \pmod 4$.
   - If $a \equiv 2 \pmod 4$, then $a^2 \equiv 2^2 \equiv 0 \pmod 4$.
   - If $a \equiv 3 \pmod 4$, then $a^2 \equiv 3^2 \equiv 1 \pmod 4$.

   Therefore, all even integers are in the same equivalence class and all odd integers are in a different equivalence class, and these are the only two equivalence classes.

4. Define the relation R on $\mathbb{R}$ by $x \, R \, y$ if $xy > 0$. Is R an equivalence relation? If so, what are the equivalence classes of R?

   Answer: No. Since $0 \cdot 0 = 0$ is not greater than 0, we know that $0 \not\!R \, 0$, so R is not reflexive.

5. Suppose R_1 and R_2 are equivalence relations on a set A. Define the relation R on A by $x\,R\,y$ if $x\,R_1\,y$ and $x\,R_2\,y$. Give the first two steps of the proof that R is an equivalence relation by showing that R is reflexive and symmetric.

Proof. **Reflexive:** Let $a \in A$. Then since R_1 and R_2 are reflexive, $a\,R_1\,a$ and $a\,R_2\,a$, so $a\,R\,a$ and R is reflexive.

Symmetric: Let $a, b \in A$ so that $a\,R\,b$. This means that $a\,R_1\,b$ and $a\,R_2\,b$. Since R_1 and R_2 are symmetric, this implies that $b\,R_1\,a$ and $b\,R_2\,a$, so $b\,R\,a$ and R is symmetric.

□

6. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Find a function $f : A \to B$ which is either injective or surjective, but not both.

Answer: Define $f : A \to B$ by the following subset of $A \times B$:

$$f = \{(1, a), (2, a), (3, b), (4, c)\}.$$

Then f is surjective since all elements of B are in the range of f: $f(1) = a$, $f(3) = b$, and $f(4) = c$. However, f is clearly not injective since $f(1) = f(2) = a$.

7. Define the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $g(m, n) = 2n - 4m$.

   (a) Is g injective? Prove or give a counterexample.

   Answer: No. Notice that $g(1, 2) = 2(2) - 4(1) = 0$ and that $g(2, 4) = 2(4) - 4(2) = 0$, so g is not injective.

   (b) Is g surjective? Prove or give a counterexample.

   Answer: No. Notice that, regardless of what m and n are, $g(m, n) = 2n - 4m = 2(n - 2m)$ is even. Therefore, since 1 is odd, there is no $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ so that $g(m, n) = 1$, and hence g is not surjective.

8. Define the function $h : \mathbb{Z}_8 \to \mathbb{Z}_8$ by

$$h([a]) = [a^3].$$

   (a) Is h injective? Prove or give a counterexample.

   Answer: No. Since $h([2]) = [2^3] = [8] = [0]$ and $h([0]) = [0^3] = [0]$, we can see that h is not injective.

   (b) Is h surjective? Prove or give a counterexample. (*Hint: Why does your answer to part (a) provide the answer to this question without doing any additional work?*)

   Answer: No. We can use the result proved in class which said that if A is a finite set and $f : A \to A$, then f is injective if and only if f is surjective. In this case, we saw in (a) that h isn't injective; since $\mathbb{Z}_8$ is finite, this means that h cannot be surjective.

9. Suppose A, B, and C are sets and that $f : A \to B$ and $g : B \to C$ are functions. If $g \circ f$ is surjective, is f necessarily surjective? Prove or give a counterexample.

Answer: No. Consider $A = B = \{1, 2\}$, $C = \{1\}$ and define $f : A \to B$ by $f(1) = f(2) = 1$ and $g : B \to C$ by $g(1) = g(2) = 1$. Then $(g \circ f)(a) = 1$ for all $a \in A$ and $g \circ f$ is obviously surjective, but f is not surjective since there is no $a \in A$ so that $f(a) = 2$.

10. Define the sequence $a_1, a_2, a_3, \ldots$ by

$$a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_n = 2a_{n-1} - a_{n-2} \text{ for all } n \geq 3.$$

Prove that $a_n = n$ for all $n \in \mathbb{N}$.

Proof. The goal is to prove this using strong induction. For $n \in \mathbb{N}$, let $P(n)$ be the statement that $a_n = n$. Then I want to show that $P(n)$ is true for all $n \in \mathbb{N}$.

Base Case: Clearly $P(1)$ and $P(2)$ are true, since $a_1 = 1$ and $a_2 = 2$.

Inductive Step: Let $k \in \mathbb{N}$ and assume $P(i)$ is true for all $1 \leq i \leq k$. In other words, we assume that $a_i = i$ whenever $1 \leq i \leq k$.

Now, the goal is to use this information to prove $P(k+1)$, which says that $a_{k+1} = k+1$. By definition,

$$a_{k+1} = 2a_k - a_{k-1}.$$

But now, by the strong inductive hypothesis, $P(k)$ and $P(k-1)$ are true, so $a_k = k$ and $a_{k-1} = k-1$. Hence,

$$a_{k+1} = 2a_k - a_{k-1} = 2(k) - (k-1) = 2k - k + 1 = k + 1,$$

and I've proved that $a_{k+1} = k+1$, so $P(k+1)$ is true.

Having proved both the base case and the (strong) inductive step, the strong principle of mathematical induction allows me to conclude that $P(n)$ is true for all $n \in \mathbb{N}$, as desired. $\square$