Math 317: Day 46

Recall:

Taylor's Theorem: Let f be defined on (a,b) w/ $a<c<b$ & assume $f^{(n)}$ exists on (a,b). Then for each $x\neq c$ in (a,b), $\exists$ y b/w c & x s.t. $R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$.

Corollary: If f is defined on (a,b) w/ $a<c<b$ & all derivatives $f^{(n)}$ exist on (a,b) & are uniformly bounded by a constant C, then $\lim_{n\to\infty} R_n(x) = 0$ $\forall$ $x\in(a,b)$.

Ex: Consider $f(x)=e^x$ on the interval $-M,M$. Then $f'(x)=e^x$ $\forall$ x (if you like, this is the definition of e^x)

So $f^{(n)}(0) = e^0 = 1$ $\forall$ n, so the Taylor series for f centered at $x=0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$.

Also, $|e^x| < e^M$ for all $x\in(-M,M)$.

Therefore, $|f^{(n)}(x)| = |e^x| \leq e^M$ for all $x\in(-M,M)$, so the Corollary implies that $\sum \frac{x^n}{n!} \to f$ on $(-M,M)$.

Proof of Taylor's Thm: The goal is to show that $\frac{f^{(n)}(y)}{n!}(x-c)^n = R_n(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k$

Now clearly there is some $M\in\mathbb{R}$ s.t. $\frac{M}{n!}(x-c)^n = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k$, so then the theorem boils down to proving that $M = f^{(n)}(y)$ for some y b/w c & x.

Since this is a statement of the form "the derivative of a function takes on some specific value on a particular interval," the natural guess is that this should be a Mean Value Theorem-type argument.

The idea is that we want some function h s.t. $h(c)=0$, $h(\tilde{x})=0$ for some $\tilde{x}$ b/w c & x, &

$h'(t) = M - f^{(n)}(t)$, b/c then Rolle's thm will imply $\exists$ $y\in(c,\tilde{x})$ s.t. $h'(y)=0$, which will imply

$f^{(n)}(y) = M$ and we'll be done.

In fact, what we're going to do is define a function g s.t. $h = g^{(n-1)}$ has the above properties. Here's g:

$$g(t) = \frac{M(t-c)^n}{n!} - R_n(t) = \frac{M(t-c)^n}{n!} - \left(f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(t-c)^k\right) = \frac{M(t-c)^n}{n!} + \sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(t-c)^k - f(t).$$

Then $\underbrace{g(c) = f(c) - f(c) = 0, \quad g(x) = 0}_{\text{Rolle's Thm implies } \exists\, x_1 \text{ b/w } c \,\&\, x \text{ s.t. } g'(x_1) = 0}$, & in fact $g^{(k)}(c) = 0$ $\forall$ $k = 0, 1, \dots, n-1$.

But now $g'(c) = 0$ & $g'(x_1) = 0 \Rightarrow \exists\, x_2$ b/w c & x_1 s.t. $g''(x_2) = 0$.

Iterate until we get x_n b/w c & x_{n-1} s.t. $g^{(n)}(x_n) = 0$

On the other hand, $g^{(n)}(t)$ is a simple function: Since $\sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(t-c)^k$ is a polynomial of degree $\leq n-1$, its n^{th} derivative is 0. Also, the n^{th} derivative of $\frac{M(t-c)^n}{n!}$ is simply $\frac{n!\,M(t-c)^0}{n!} = M$, so

$$g^{(n)}(t) = M - f^{(n)}(t).$$

But now we've succeeded: $0 = g^{(n)}(x_n) = M - f^{(n)}(x_n)$, so $f^{(n)}(x_n) = M$ & we've succeeded in finding $y = x_n$ b/w c & x s.t. $f^{(n)}(y) = M$.

▨