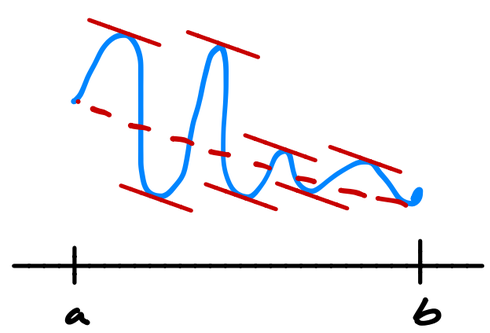
Math 317: Day 42

Recall from calculus that if a differentiable function f attains a max or min at x_0 which is in the interior of an interval on which f is defined, then $f'(x_0)=0$. More precisely:

Thm: Suppose f is differentiable on an open interval containing x_0. If f has a (local) max or min. at x_0, then $f'(x_0)=0$.

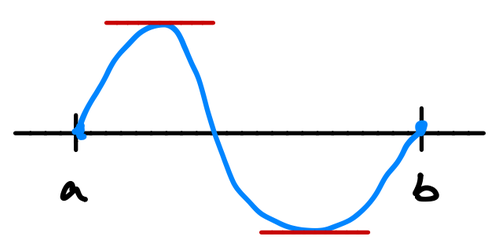
This is a key piece in the proof of

Mean Value Theorem: Let f be continuous on $[a,b]$ & differentiable on (a,b). Then there exists $x_0\in(a,b)$ so that $f'(x_0)=\frac{f(b)-f(a)}{b-a}$.



Special case which we'll prove first:

Rolle's Thm: Let f be continuous on $[a,b]$ & differentiable on (a,b) so that $f(a)=f(b)$. Then $\exists\, x_0\in(a,b)$ s.t. $f'(x_0)=0$.



Why does MVT follow from Rolle's Thm?

Proof of MVT: Let $h(x)=f(x)-\left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$. Then $h'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$, so h is diff'ble wherever f is, & $h(a)=0=h(b)$, so

Rolle's Thm $\Rightarrow \exists\, x_0\in(a,b)$ s.t. $0=h'(x_0)=f'(x_0)-\frac{f(b)-f(a)}{b-a} \Rightarrow f'(x_0)=\frac{f(b)-f(a)}{b-a}$. ∎

Proof of Rolle's Thm: Since f is continuous on $[a,b]$, it attains its max & min; i.e., $\exists\, x_1,x_2\in[a,b]$ s.t. $f(x_1)\le f(x)\le f(x_2)$

for all $x\in[a,b]$.

If x_1 & x_2 are both endpoints, then $f(x_1)=f(x_2)$, so f is constant on $[a,b]$, which implies $f'(x)=0$ $\forall\, x\in(a,b)$.

Otherwise, f has a max or a min at some $x_0\in(a,b)$, & then $f'(x_0)=0$. ∎