Math 317: Day 28

We are moving ever closer to the purported topic of this course, namely calculus.

Eventually we are going to define the derivative of a function f at a to be $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, so we really need to say what this $\lim_{x\to a}$ business means. In the derivative, if we let $g(x) = \frac{f(x)-f(a)}{x-a}$, we're just taking $\lim_{x\to a} g(x)$, which we now define:

Def: Let $S \subseteq \mathbb{R}$, let $a \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ so that a is the limit of at least one sequence in S, let $L \in \mathbb{R} \cup \{\pm\infty\}$, & let f be a function defined on S. Then we write $\lim_{x\to a^S} f(x) = L$ if for any sequence (x_n) in S w/ $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = L$.

Ex: f is cont. at x_0 & $S = \text{domain}(f) \iff \lim_{x\to a^S} f(x) = f(a)$.

We write just $\lim_{x\to a} f(x) = L$ if there is a set $S = (c,a) \cup (a,b)$ for $c < a < b$ s.t. $\lim_{x\to a^S} f(x) = L$.

Similarly, we'll write $\lim_{x\to a^+} f(x) = L$ if $\lim_{x\to a^S} f(x) = L$ for some $S = (a,b)$, & we'll write $\lim_{x\to a^-} f(x) = L$ if $\lim_{x\to a^S} f(x) = L$ for some $S = (c,a)$.

Ex: Let's think about what happens when we want to compute $f'(-3)$ when $f(x) = x^2$. Then we're interested in

$$\lim_{x\to -3} \frac{f(x)-f(-3)}{x-(-3)} = \lim_{x\to -3} \frac{x^2-(-3)^2}{x+3} = \lim_{x\to -3} \frac{x^2-9}{x+3}.$$

So let $g(x) = \frac{x^2-9}{x+3}$. Clearly $g(-3)$ doesn't make sense, so we can't just plug in $x=-3$.

Now, I claim $\lim_{x\to -3} g(x) = -6$, even tho $g(-3)$ is undefined.

To see this, let $S = (-\infty,-3) \cup (3,\infty)$. Then for any sequence (x_n) in S converging to -3 we have

$$g(x_n) = \frac{x_n^2-9}{x_n+3} = \frac{(x_n-3)(x_n+3)}{x_n+3} = x_n - 3 \quad \& \text{ of course } \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(x_n-3) = \left(\lim_{n\to\infty} x_n\right) - 3 = -3-3 = -6,$$

so $\lim_{x\to -3^S} g(x) = -6$. Since S is of the form specified above, we see that $\lim_{x\to -3} g(x) = -6$.

In other words, we've just proved that $f'(-3) = -6$, which probably shouldn't be too surprising.

Equivalently to how we defined $\lim_{x \to a^+} f(x)$:

Thm: Let f be defined on $S \subseteq \mathbb{R}$ & let a be a limit point of S & L a number. Then $\lim_{x \to a^+} f(x) = L$ if & only if

for all $\varepsilon > 0$ there exists $\delta > 0$ so that $x \in S$ & $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$.