Math 317 Final Exam Practice Problem Solutions

1. For each of the following statements, say whether it is true or false. If the statement is true give a brief explanation why (this does not need to be a rigorous proof); if it is false, give a counterexample.

   (a) The set $S = \{\frac{p}{q} : p \text{ and } q \text{ integers}, q > 20\}$ is bounded below.
   Answer: False. The numbers $-\frac{n}{25}$ are in S for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} -\frac{n}{25} = -\infty$, so S can't be bounded below.

   (b) The sequence (s_n) with $s_n = \frac{n}{3^n}$ for all n is a convergent sequence.
   Answer: True. This is a monotone decreasing sequence which is bounded below by 0, so the Monotone Convergence Theorem implies that it converges (in fact, it converges to 0). To see that it is monotone, notice that for any n

   $$a_n - a_{n+1} = \frac{n}{3^n} - \frac{n+1}{3^{n+1}} = \frac{3n}{3^{n+1}} - \frac{n+1}{3^{n+1}} = \frac{2n-1}{3^{n+1}} > 0$$

   so $a_n > a_{n+1}$ for all n, and hence the sequence is decreasing. The fact that it's bounded below by 0 follows from the fact that the numerator and denominator are both positive for all n.

   (c) The equation $\cos(x) = \tan(x)$ has a solution in $[0, \pi/4]$. (It may be useful to recall that $\frac{1}{\sqrt{2}} < 1$.)
   Answer: True. The idea is the use the Intermediate Value Theorem, which means we need a single continuous function on $[0, \pi/4]$. To that end, define $f(x) = \cos(x) - \tan(x)$. Since $\cos(x)$ and $\tan(x)$ are both continuous on $[0, \pi/4]$ (discontinuities of $\tan(x)$ only occur at odd multiples of $\pi/2$), we see that f is continuous. Also,

   $$f(0) = \cos(0) - \tan(0) = 1 - 0 = 1 > 0$$
   $$f(\pi/4) = \cos(\pi/4) - \tan(\pi/4) = \frac{1}{\sqrt{2}} - 1 < 0.$$

   So f is a continuous function on $[0, \pi/4]$ with $f(0) > 0$ and $f(\pi/4) < 0$, so the Intermediate Value Theorem guarantees the existence of $x_0 \in (0, \pi/4)$ so that $f(x_0) = 0$, which implies that $\cos(x_0) = \tan(x_0)$.

   (d) If a function f has a maximum at $c \in \mathbb{R}$, then f is differentiable at c.
   Answer: False. Consider $f(x) = -|x|$. Then f has a maximum value of 0 at $x = 0$, but is not differentiable at $x = 0$.

   (e) Suppose f and g are differentiable on all of $\mathbb{R}$. Then the function

   $$h(x) = (f(x))^2 - 3g(x)$$

   is also differentiable on all of $\mathbb{R}$.
   Answer: True. Since f is differentiable, so is f^2; likewise, since g is differentiable, so is $-3g$. Now $h(x) = (f(x))^2 - 3g(x)$ is the sum of two differentiable functions, and hence must be differentiable. In fact, using the Chain Rule we can compute

   $$h'(x) = 2f(x)f'(x) - 3g'(x).$$

(f) Suppose g is integrable on $[a,b]$ and that there exists $c \in (a,b)$ such that

$$\int_a^c g > \int_a^b g.$$

Then there exists some point $d \in (a,b)$ such that $g(d) < 0$.

Answer: True. We know that

$$\int_a^b g = \int_a^c g + \int_c^b g,$$

so

$$\int_c^b g = \int_a^b g - \int_a^c g < 0.$$

But this can only happen if g is negative somewhere between c and b, so there exists some $d \in (c,d)$ so that $g(d) < 0$.

(g) If $h_n(x) = x - x^n$, then (h_n) converges uniformly on $[0,1]$.

Answer: False. For any $x \in [0,1)$, we know that $x^n \to 0$ as $n \to \infty$, whereas obviously $1^n = 1$ for all n. Therefore

$$\lim_{n\to\infty} h_n(x) = \begin{cases} x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

This is a well-defined function, but it is not continuous, so it can't be the uniform limit of continuous functions. The h_n's are all continuous, so they can't be converging uniformly.

(h) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$.

Answer: True. For any n we have

$$|a_n \sin(nx)| = |a_n||\sin(nx)| \leq |a_n|$$

since $|\sin(y)| \leq 1$ for any real number y. Therefore, since $\sum |a_n|$ converges, the Weierstrass M-test implies that the series $\sum a_n \sin(nx)$ converges uniformly.

(i) Every power series converges on some interval (a,b) with $a \neq b$.

Answer: False. Consider the power series

$$\sum_{n=0}^{\infty} n! x^n.$$

Then

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} n+1 = \infty,$$

so the Ratio Test implies that this power series can *only* converge at $x = 0$, and hence there is no interval (a,b) with $a \neq b$ on which it converges.

2. Give an example of each of the following, or explain why no such example can exist. Such an explanation need not be a completely rigorous proof, but it should be clear and convincing and should cite any relevant theorems or definitions.

(a) A bounded set which is not an interval.

Answer: Let $A = \{0, 1\}$. This is definitely not an interval, since it does not contain infinitely many points, but it is bounded. Indeed, any finite set would be an example of a bounded set which is not an interval. Of course, there are also examples which are infinite sets; for example, the set of all rational numbers between 0 and 1 is bounded and not an interval.

(b) An unbounded sequence (a_n) with $\liminf a_n = 0$.

Answer: Let (a_n) be the sequence $(1, 0, 2, 0, 3, 0, 4, 0, \ldots)$. In other words

$$a_n = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Then (a_n) is definitely unbounded, since the odd terms diverge to infinity, but

$$\liminf a_n = \lim_{N \to \infty} \inf\{a_n : n > N\} = 0,$$

since every set $\{a_n : n > N\}$ contains 0.

(c) A divergent sequence (s_n) with a convergent subsequence.

Answer: The sequence from part (b) works. This is definitely a divergent sequence, but the subsequence $(0, 0, \ldots)$ converges to 0.

(d) A continuous function on $[0, 1]$ which is not uniformly continuous.

Answer: No such example can exist. A continuous function on a closed interval must be uniformly continuous.

(e) A continuous function on $(0, 1)$ which is not uniformly continuous.

Answer: Let $f(x) = \frac{1}{x}$. Then f is continuous on $(0, 1)$, but not uniformly continuous, which we can see by observing that f cannot be extended to a continuous function on $[0, 1]$ (there's no way to assign a value at $x = 0$ that makes f continuous). We can also prove this by contradiction. Suppose f were uniformly continuous on $(0, 1)$. Then there would exist some δ so that $|x - y| < \delta$ would imply that $|f(x) - f(y)| < 1$. But now choose $k > \frac{2}{\delta}$ and let $x = \frac{1}{k}$ and $y = \frac{1}{k+1}$

$$|x - y| = \left|\frac{1}{k} - \frac{1}{k+1}\right| \le \left|\frac{1}{k}\right| + \left|\frac{1}{k+1}\right| \le \left|\frac{1}{k}\right| + \left|\frac{1}{k}\right| = \frac{2}{k} < \frac{2}{\frac{2}{\delta}} = \delta,$$

but

$$|f(x) - f(y)| = |k - (k+1)| = |-1| = 1,$$

which is not less than 1. From this contradiction, then, we see that f cannot be uniformly continuous on $(0, 1)$.

(f) A differentiable function f defined on $[0, 1]$ that is not integrable.

Answer: No such example can exist. All differentiable functions are continuous, and we know that continuous functions are integrable.

3. Determine $\lim_{n\to\infty} a_n$ where the terms of the sequence (a_n) are given by

$$a_n = \frac{3n+4}{7n}.$$

Answer: The limit of the sequence is $3/7$. To prove this, let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \frac{4}{7\epsilon}$. Then, for any $n \geq N$,

$$n \geq N > \frac{4}{7\epsilon},$$

so

$$\left|a_n - \frac{3}{7}\right| = \left|\frac{3n+4}{7n} - \frac{3}{7}\right| = \left|\frac{3n+4}{7n} - \frac{3n}{7n}\right| = \left|\frac{4}{7n}\right| = \frac{4}{7n} < \frac{4}{7\frac{4}{7\epsilon}} = \epsilon.$$

Since our choice of $\epsilon > 0$ was arbitrary, we see that $\lim a_n = 3/7$.

4. Determine which of the following sequences and series converge. Briefly justify your answer.

   (a) $\left(1 - \frac{2}{n^2}\right)$
   Answer: This sequence converges. We know that

$$\lim_{n\to\infty} \frac{2}{n^2} = 2 \lim_{n\to\infty} \frac{1}{n^2} = 2 \times 0 = 0,$$

   so

$$\lim_{n\to\infty} \left(1 - \frac{2}{n^2}\right) = 1 - \lim_{n\to\infty} \frac{2}{n^2} = 1 - 0 = 1.$$

   (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$
   Answer: This series diverges. We can see this since the terms are of the form $\frac{1}{k^p}$ with $p = \frac{1}{2} < 1$. We could also use the Integral Test:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} 2\sqrt{x}\Big]_0^b = \lim_{b\to\infty} \left(2\sqrt{b} - 2\sqrt{0}\right) = +\infty,$$

   so the Integral Test says that the series diverges.

   (c) $\sum_{k=0}^{\infty} \frac{2^k}{k!}$
   Answer: This series converges by the Ratio Test. We compute

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}}\right| = \lim_{k\to\infty} \frac{2}{k+1} = 0.$$

   Since this is less than 1, the Ratio Test implies the series converges.

5. For each of the following, determine whether or not the sequence or series converges on the given domain. You don't need to give a complete proof, but you should justify your answer.

(a) The sequence (g_n) on $(0, 1)$, where $g_n(x) = \frac{x}{nx+1}$.
Answer: The sequence converges. Notice that dividing numerator and denominator by x yields

$$\frac{x}{nx+1} = \frac{1}{n+1/x} < \frac{1}{n}.$$

On the other hand, each term is greater than zero, so we have

$$0 < \frac{x}{nx+1} < \frac{1}{n}.$$

Since $\frac{1}{n} \to 0$, this can only happen if $\lim_{n\to\infty} \frac{x}{nx+1} = 0$.

(b) The series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$, where

$$f_n(x) = \begin{cases} 0 & \text{if } x \le n \\ (-1)^n & \text{if } x > n. \end{cases}$$

Answer: The series converges. To see this, let $x \in \mathbb{R}$. Then for any $n \ge x$, we have that $f_n(x) = 0$, so only finitely many terms in the series

$$\sum_{n=1}^{\infty} f_n(x)$$

are nonzero. Since finite sums are certainly well-defined, the above series must converge. Since the choice of x was arbitrary, we see that $\sum f_n$ converges on all of $\mathbb{R}$ (though definitely not uniformly).

6. Use the definition of integrability to show that $f(x) = 4$ is integrable on $[0, 1]$.

Proof. Notice that on any subinterval $[a, b]$ of $[0, 1]$, we have

$$M(f, [a, b]) = 4 \quad \text{and} \quad m(f, [a, b]) = 4.$$

If $P = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ is a partition of $[0, 1]$, then

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M(f, [t_{i-1}, t_i])(t_i - t_{i-1}) = \sum_{i=1}^{n} 4(t_i - t_{i-1}) \\ &= 4\left[(t_1 - t_0) + (t_2 - t_1) + \ldots + (t_n - t_{n-1})\right] = 4(t_n - t_0) = 4(1 - 0) = 4 \end{aligned}$$

and

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m(f, [t_{i-1}, t_i])(t_i - t_{i-1}) = \sum_{i=1}^{n} 4(t_i - t_{i-1}) \\ &= 4\left[(t_1 - t_0) + (t_2 - t_1) + \ldots + (t_n - t_{n-1})\right] = 4(t_n - t_0) = 4(1 - 0) = 4 \end{aligned}$$

Therefore, $U(f, P) = 4 = L(f, P)$ for all partitions P, so $U(f) = 4 = L(f)$, and hence f is integrable on $[0, 1]$ with $\int_0^1 f = 4$. □