Math 317 Final Exam Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true give a brief explanation why (this does not need to be a rigorous proof); if it is false, give a counterexample.

   (a) The set $S = \{\frac{p}{q} : p \text{ and } q \text{ integers}, q > 20\}$ is bounded below.

   (b) The sequence (s_n) with $s_n = \frac{n}{3^n}$ for all n is a convergent sequence.

   (c) The equation $\cos(x) = \tan(x)$ has a solution in $[0, \pi/4]$. (It may be useful to recall that $\frac{1}{\sqrt{2}} < 1$.)

   (d) If a function f has a maximum at $c \in \mathbb{R}$, then f is differentiable at c.

   (e) Suppose f and g are differentiable on all of $\mathbb{R}$. Then the function
   $$h(x) = (f(x))^2 - 3g(x)$$
   is also differentiable on all of $\mathbb{R}$.

   (f) Suppose g is integrable on $[a, b]$ and that there exists $c \in (a, b)$ such that
   $$\int_a^c g > \int_a^b g.$$
   Then there exists some point $d \in (a, b)$ such that $g(d) < 0$.

   (g) If $h_n(x) = x - x^n$, then (h_n) converges uniformly on $[0, 1]$.

   (h) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$.

   (i) Every power series converges on some interval (a, b) with $a \neq b$.

2. Give an example of each of the following, or explain why no such example can exist. Such an explanation need not be a completely rigorous proof, but it should be clear and convincing and should cite any relevant theorems or definitions.

   (a) A bounded set which is not an interval.

   (b) An unbounded sequence (a_n) with $\liminf a_n = 0$.

   (c) A divergent sequence (s_n) with a convergent subsequence.

   (d) A continuous function on $[0, 1]$ which is not uniformly continuous.

   (e) A continuous function on $(0, 1)$ which is not uniformly continuous.

   (f) A differentiable function f defined on $[0, 1]$ that is not integrable.

3. Determine $\lim_{n\to\infty} a_n$ where the terms of the sequence (a_n) are given by
   $$a_n = \frac{3n + 4}{7n}.$$

4. Determine which of the following sequences and series converge. Briefly justify your answer.

   (a) $\left(1 - \frac{2}{n^2}\right)$

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

(c) $\sum_{k=0}^{\infty} \frac{2^k}{k!}$

5. For each of the following, determine whether or not the sequence or series converges on the given domain. You don't need to give a complete proof, but you should justify your answer.

   (a) The sequence (g_n) on $(0,1)$, where $g_n(x) = \frac{x}{nx+1}$.

   (b) The series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$, where

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq n \\ (-1)^n & \text{if } x > n. \end{cases}$$

6. Use the definition of integrability to show that $f(x) = 4$ is integrable on $[0,1]$.