Math 317 Exam #2 Practice Problem Solutions

1. For each of the following statements, say whether it is true or false. If the statement is true, explain why; if it is false, give a counterexample.

   (a) The equation $2x^{17} - 4x^{12} + 1 = 5x^{23} + 10x^8 - 2$ has a solution in the interval $[0, 1]$.
   Answer: True. To prove this, let $f(x) = 5x^{23} - 2x^{17} + 4x^{12} + 10x^8 - 3$. Then we know that the equation has a solution if and only if $f(x) = 0$. But f is a polynomial and hence a continuous function, and we can compute

   $$f(0) = -3 \qquad \text{and} \qquad f(1) = 14.$$

   Hence, since $f(0) < 0 < f(1)$, the Intermediate Value Theorem implies that there exists some $x_0 \in (0, 1)$ so that $f(x_0) = 0$. Then x_0 is a solution of the given equation.

   (b) Every series of the form $\sum(-1)^n a_n$ converges.
   Answer: False. Consider the series of this form with each $a_n = 1$, namely $\sum(-1)^n$. This series diverges.

   (c) The series $\sum_{n=1}^{\infty} \frac{\cos^n x}{n^2}$ converges uniformly on all of $\mathbb{R}$.
   Answer: True. Notice that for each n,

   $$\left|\frac{\cos^n x}{n^2}\right| = \frac{|\cos^n x|}{n^2} \leq \frac{1}{n^2}$$

   for all $x \in \mathbb{R}$.
   But now we know that the series $\sum \frac{1}{n^2}$ converges, so the Weierstrass M-test tells us that the given series converges uniformly on all of $\mathbb{R}$.

2. Give examples of each of the following, or explain why no such example exists.

   (a) A uniformly continuous function on the interval $(0, 1)$.
   Answer: The function $f(x) = 0$ is uniformly continuous on $(0, 1)$. To prove this rigorously, let $\epsilon > 0$. If we choose $\delta = 1$, then for any $x, y \in (0, 1)$ with $|x - y| < 1$, we have

   $$|f(x) - f(y)| = |0 - 0| = 0 < \epsilon.$$

   Since the choice of $\epsilon > 0$ is arbitrary, we have shown that f is uniformly continuous on $(0, 1)$.

   (b) A continuous function with domain the interval $[0, 1]$ and range the interval $(0, 1)$.
   Answer: This is impossible. Since f is continuous on $[0, 1]$, we know by Theorem 18.1 that f attains its maximum and minimum values $M = \max\{f(x) : x \in [0, 1]\}$ and $m = \min\{f(x) : x \in [0, 1]\}$. In other words, the range of f must be the closed interval $[m, M]$.

   (c) A sequence (f_n) of continuous functions which converges uniformly to a continuous function f on some set S.
   Answer: You proved in HW 7 that $f_n(x) = (x - \frac{1}{n})^2$ converges uniformly to $f(x) = x^2$ on $S = [0, 1]$.
   A more mundane example: $f_n(x) = 0$ converges uniformly to $f(x) = 0$ on all of $\mathbb{R}$.

(d) A power series with interval of convergence $(-2, 2)$.

Answer: Consider the power series $\sum \frac{x^n}{2^n}$. Using the Root Test we have

$$\beta = \limsup \left| \frac{1}{2^n} \right|^{1/n} = \limsup \frac{1}{2} = \frac{1}{2},$$

so the radius of convergence of the series is $R = \frac{1}{\beta} = 2$. When we check the endpoints, we see that, for $x = 2$ the series is

$$\sum \frac{2^n}{2^n} = \sum 1,$$

which diverges, and for $x = -2$ the series is

$$\sum \frac{(-2)^n}{2^n} = \sum (-1)^n,$$

which diverges. Therefore, the interval of convergence is exactly $(-2, 2)$.

3. (a) Give a rigorous definition of

$$\lim_{x \to +\infty} f(x) = L.$$

Answer: By definition, $\lim_{x\to+\infty} f(x) = L$ if for all sequence (x_n) diverging to $+\infty$ we have $\lim f(x_n) = L$.

(b) Using your definition from part (a), show that

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

Proof. Let (x_n) be a sequence diverging to $+\infty$. Let $\epsilon > 0$. Then, by definition of $\lim x_n = +\infty$, there exists N so that $n > N$ implies that $x_n > \frac{1}{\sqrt{\epsilon}}$.[1] But then for any $n > N$ we have

$$|f(x_n) - 0| = |f(x_n)| = \left| \frac{1}{x_n^2} \right| = \frac{1}{x_n^2} < \frac{1}{(1/\sqrt{\epsilon})^2} = \frac{1}{1/\epsilon} = \epsilon.$$

But since $\epsilon > 0$ was chosen arbitrarily, we see that such an N exists for all ϵ, so $\lim f(x_n) = L$.

In turn, our sequence (x_n) with $\lim x_n = +\infty$ was chosen arbitrarily, so we see that $\lim f(x_n) = L$ for all such (x_n), which is our definition for $\lim_{x\to+\infty} f(x) = L$. □

4. Determine the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} x^n.$$

[1] The following argument goes slightly astray if $\epsilon > 1$, so if ϵ happens to be larger than 1, just choose $x_n > 1$ instead, which will imply $\frac{1}{x_n^2} < \frac{1}{1} = 1 < \epsilon$.

Answer: Using the Ratio Test, we have

$$\beta = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{\sqrt{n+1}}}{\frac{2^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{\sqrt{n}}{\sqrt{n+1}} = 2 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 2 \cdot 1 = 2.$$

Therefore, the radius of convergence of the series is $R = \frac{1}{\beta} = \frac{1}{2}$. Now check the endpoints.

When $x = 1/2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges.

When $x = -1/2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which converges by the Alternating Series Test.

Therefore, the interval of convergence of the series is $[-1/2, 1/2)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function defined on all of $\mathbb{R}$. Prove that $\lim_{x\to c^+} f(x)$ exists for every $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$. Consider the set

$$A = \{f(x) : c < x\}.$$

Then, since f is increasing, $f(c)$ is a lower bound for A. Hence, by the Axiom of Completeness, A has an infimum $L = \inf A$. I claim that $\lim_{x\to c^+} f(x) = L$.

To see this, let $\epsilon > 0$. Then, $L + \epsilon$ is not a lower bound for A, so there exists $f(x_0) \in A$ such that $f(x_0) < L + \epsilon$. Let $\delta = x_0 - c$. Whenever $0 < x - c < \delta$ we have that

$$0 < f(x) - L < f(x_0) - L < \epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, this suffices to show that $\lim_{x\to c^+} f(x) = L$. □