Math 317 Exam #1 Practice Problem Solutions

1. For each of the following statements, say whether it is true or false. If the statement is true give a brief explanation why (this does not need to be a proof); if it is false, give a counterexample.

   (a) Every bounded set contains its supremum.

   Answer. False. Consider the interval $A = (0, 1)$. A is clearly bounded above and below, as 0 is a lower bound and 1 is an upper bound. Moreover, $1 = \sup A$, but $1 \notin A$, so A is an example of a bounded set that does not contain its supremum.

   (b) A decreasing sequence is always bounded.

   Answer. False. The sequence $(0, -1, -2, -3, \ldots)$ is a decreasing sequence that is unbounded (since it has no lower bound).

   (c) Every Cauchy sequence is bounded.

   Answer. True. Every Cauchy sequence converges (Theorem 10.11) and every convergent sequence is bounded (Theorem 9.1). Alternatively, Lemma 10.10 says Cauchy sequences are bounded.

   (d) Every sequence contains a convergent subsequence.

   Answer. False. The sequence $(1, 2, 3, \ldots)$ contains no convergent subsequences since every subsequence is unbounded.

2. Give examples of each of the following, or explain why no such example exists (this explanation does not need to be a proof, but you should give a clear and convincing explanation).

   (a) An alternating sequence that converges. (An *alternating sequence* is one whose terms alternate between positive and negative.)

   Answer. The sequence (a_n) given by

   $$a_n = \frac{(-1)^n}{n}$$

   is an alternating sequence that converges to zero.

   (b) A bounded, divergent sequence.

   Answer. The sequence (a_n) given by

   $$a_n = (-1)^n$$

   is a bounded sequence that diverges.

3. Determine the following numbers (don't worry about proofs; just find the answer any way you like).

   (a) The infimum of the set

   $$\{x \in \mathbb{R} : -1 < 2x - x^2\}.$$

   Answer. If x is in the given set, $-1 < 2x - x^2$ or, equivalently,

   $$-x^2 + 2x + 1 > 0.$$

Hence, the extrema of the set are given by those values of x such that $-x^2+2x+1=0$. Using the quadratic formula, the extrema are

$$x = \frac{-2 \pm \sqrt{4-4(-1)(1)}}{2(-1)} = \frac{-2 \pm \sqrt{8}}{-2} = 1 \mp \sqrt{2}.$$

The smaller of the two is the infimum, so we see that the infimum of the set is $1-\sqrt{2}$.

(b) $\lim_{n\to\infty} a_n$ where the terms of the sequence (a_n) are given by

$$a_n = \frac{2n-3}{5n^2+4}.$$

Answer. The limit of the sequence is 0. For a rigorous proof using the definition (which you didn't need to give), let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \frac{2}{5\epsilon}$. Then, for any $n > N$ we have $n > N > \frac{2}{5\epsilon}$, so

$$|a_n - 0| = \left|\frac{2n-3}{5n^2+4} - 0\right| = \left|\frac{2n-3}{5n^2+4}\right| = \frac{|2n-3|}{5n^2+4} \le \frac{|2n-3|}{5n^2} \le \frac{2n}{5n^2} = \frac{2}{5n} < \frac{2}{5\frac{2}{5\epsilon}} = \epsilon.$$

Since our choice of $\epsilon > 0$ was arbitrary, we see that $\lim a_n = 0$.

Alternatively, I could also have proved this using the Algebraic Limit Theorem as follows: multiply each a_n by $\frac{1/n^2}{1/n^2}$ to see that

$$a_n = \frac{2n-3}{5n^2-4}\frac{1/n^2}{1/n^2} = \frac{2/n - 3/n^2}{5 - 4/n}.$$

But now, repeatedly applying the Algebraic Limit Theorem gives

$$\lim_{n\to\infty} a_n = \frac{\lim(2/n - 3/n^2)}{\lim(5 - 4/n)} = \frac{2\lim 1/n - 3\lim 1/n^2}{5 - 4\lim 1/n} = \frac{0-0}{5-0} = 0.$$

4. A number t is called a *cluster point* of the sequence (s_n) if for every $\epsilon > 0$ there are infinitely many values of n with $|s_n - t| < \epsilon$.

(a) Prove that if there is a subsequence of (s_n) that converges to t, then t is a cluster point of (s_n).

Proof. Let (s_{n_k}) be the subsequence of (s_n) that converges to t. Let $\epsilon > 0$. Then, by the definition of convergence, there exists $N \in \mathbb{N}$ such that, for all $k \ge N$,

$$|s_{n_k} - t| < \epsilon.$$

Since there are infinitely many $k \ge N$, we see that t is a cluster point of (s_n). □

(b) True or false: every bounded sequence has a cluster point. If the statement is true, prove it. If it is false, find an example of a bounded sequence with no cluster point.

Answer: True. By the Bolzano–Weierstrass Theorem, every bounded sequence has a convergent subsequence. But then by part (a), the limit of the convergent subsequence is a cluster point of the sequence.