Math 317 Final Exam

Due 3:00 PM Friday, December 17.

Name: ____

You may refer to your notes, your text, your homework, or the homework solutions. Do not discuss the questions with anyone except me.

There is no time limit, and the exam is due at 3:00 PM on Friday, December 17. If you plan to ask me questions about course content, please do not look at the problems until after you do so.

Please do not sign the following until the examination is completed.

I accept full responsibility under the Haverford Honor Code for my conduct on this examination.

Signed: ____

1. Show that any collection of disjoint nonempty open sets of real numbers is countable or finite.

2. A set $A \subseteq \mathbb{R}$ is called *relatively compact* if its closure $\overline{A}$ is compact. Prove that A is relatively compact if and only if every sequence in A has a subsequence that converges to a point in $\mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and bounded above. Prove that the limit $\lim_{x \to c^+} f(x)$ exists for every $c \in \mathbb{R}$.

   (*Comment/Definition*: $\lim_{x \to c^+} f(x) = L$ means that for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - c < \delta$ implies $|f(x) - L| < \epsilon$.)

4. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous, f is differentiable on $(0, 1)$, and $f(0) = 0$. Assume that $|f'(x)| \leq |f(x)|$ for all $x \in (0, 1)$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

5. Prove the following Mean Value Theorem for integrals: If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

   $$f(c) = \frac{1}{b-a} \int_a^b f.$$

   The quantity $\frac{1}{b-a} \int_a^b f$ is called the *average value* of f on $[a, b]$.