Math 2260 Exam #3 Solutions

1. Consider the sequence $(a_n)_{n=1}^{\infty}$ where

$$a_n = \left(n^2+n\right)^{1/n}.$$

Does this sequence converge or diverge? If it converges, find its limit.

Answer: Notice that the term inside parentheses is going to ∞, whereas the power is going to zero, so this is an indeterminate form. To deal with it, first find the limit of the sequence of natural logarithms:

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\left(n^2+n\right)^{1/n}\right) &= \lim_{n\to\infty} \frac{1}{n}\ln\left(n^2+n\right)\\ &= \lim_{n\to\infty} \frac{\ln\left(n^2+n\right)}{n}\\ &= \lim_{x\to\infty} \frac{\ln\left(x^2+x\right)}{x}\\ &= \lim_{x\to\infty} \frac{\frac{1}{x^2+x}(2x+1)}{1}\\ &= \lim_{x\to\infty} \frac{2x+1}{x^2+x}\\ &= 0\end{aligned}$$

where I went from the third to the fourth lines using L'Hôpital's Rule.

Therefore,

$$\lim_{n\to\infty} \left(n^2+n\right)^{1/n} = e^0 = 1,$$

so the sequence converges to 1.

2. Does the series

$$\sum_{n=1}^{\infty} 7^{n+1}\frac{(-1)^n}{6^{n-1}}$$

converge or diverge? If it converges, find its sum. If it diverges, explain why.

Answer: Since everything in sight is being raised to a power involving n, this looks like geometric series. To confirm that intuition, write out the first few terms:

$$\sum_{n=1}^{\infty} 7^{n+1}\frac{(-1)^n}{6^{n-1}} = -7^2 + \frac{7^3}{6} - \frac{7^4}{6^2} + \frac{7^5}{6^3} - \dots.$$

Now factor -7^2 out of this sum:

$$-7^2\left(1 - \frac{7}{6} + \frac{7^2}{6^2} - \frac{7^3}{6^3} + \dots\right).$$

Hence, this is a geometric series with $a = -7^2 = -49$ and $r = -\frac{7}{6}$. Since

$$|r| = \left|-\frac{7}{6}\right| = \frac{7}{6} > 1,$$

the series must diverge.

3. For what values of p does the series

$$\sum_{n=1}^{\infty} \frac{n^p}{3+n^3}$$

converge? Explain your answer.

Answer: This series looks quite similar to the series

$$\sum_{n=1}^{\infty} \frac{n^p}{n^3}.$$

To make "looks quite similar to" more precise, I'll do a limit comparison:

$$\lim_{n\to\infty} \frac{\frac{n^p}{3+n^3}}{\frac{n^p}{n^3}} = \lim_{n\to\infty} \frac{n^p}{3+n^3} \cdot \frac{n^3}{n^p} = \lim_{n\to\infty} \frac{3+n^3}{n^3} = 1.$$

Therefore, the given series will converge precisely when the series

$$\sum_{n=1}^{\infty} \frac{n^p}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{3-p}}$$

converges. This series will converge precisely when the power on n is greater than 1, meaning when $3-p>1$ or $p<2$.

To demonstrate this rigorously, I can use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^{3-p}}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^{3-p}}\, dx = \lim_{b\to\infty} \int_1^b x^{p-3}\, dx.$$

So long as $p-3 \neq -1$, meaning $p \neq 2$, this is equal to

$$\begin{aligned}\lim_{b\to\infty} \left[\frac{x^{p-3+1}}{p-3+1}\right]_1^b &= \lim_{b\to\infty} \left[\frac{x^{p-2}}{p-2}\right]_1^b \\ &= \lim_{b\to\infty} \left(\frac{b^{p-2}}{p-2} - \frac{1}{p-2}\right),\end{aligned}$$

which converges to $\frac{1}{2-p}$ when $p<2$ and diverges when $p>2$.

Finally, when $p=2$, the integral above is equal to

$$\lim_{b\to\infty} [\ln(x)]_1^b = \lim_{b\to\infty} (\ln(b) - \ln(1)) = \infty.$$

Therefore, the integral, and hence the series, converges precisely when $p<2$.

4. What is the interval of convergence of the following power series? Explain your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n3^n}.$$

Answer: Use the Ratio Test on the series of absolute values to get started:

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)3^{n+1}}\right|}{\left|\frac{(-1)^n(x-2)^n}{n3^n}\right|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x-2|^n} = \lim_{n\to\infty} \frac{|x-2|}{3} \cdot \frac{n}{n+1} = \frac{|x-2|}{3}.$$

Therefore, the given series converges absolutely when $\frac{|x-2|}{3} < 1$, meaning when $|x-2| < 3$.

Now, check the endpoints. When $x - 2 = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which satisfies the hypotheses of the Alternating Series Test and hence converges.

On the other hand, when $x - 2 = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges.

Therefore, the series converges for

$$-3 < x - 2 \leq 3,$$

or, adding 2 to each of the three terms,

$$-1 < x \leq 5,$$

so the interval of convergence is $(-1, 5]$.

5. Find the Taylor series centered at $x = \pi$ for the function

$$f(x) = \sin(2x).$$

(It suffices to write down, say, the first three non-zero terms, but it's even better to give the general form.)

Answer: Remember that the Taylor series for $f(x)$ centered at $x = \pi$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n,$$

so we need to know the values of the derivatives of f at $x = \pi$. First, I compute the derivatives:

$$\begin{aligned} f'(x) &= 2\cos(2x) \\ f''(x) &= -4\sin(2x) \\ f'''(x) &= -8\cos(2x) \\ f^{(4)}(x) &= 16\sin(2x) \\ f^{(5)}(x) &= 32\cos(2x) \\ &\vdots \end{aligned}$$

Therefore, the values of f and its derivatives at $x = \pi$ are

$$\begin{aligned} f(\pi) &= \sin(2\pi) = 0 \\ f'(\pi) &= 2\cos(2\pi) = 2 \\ f''(\pi) &= -4\sin(2\pi) = 0 \\ f'''(\pi) &= -8\cos(2\pi) = -8 \\ f^{(4)}(\pi) &= 16\sin(2\pi) = 0 \\ f^{(5)}(\pi) &= 32\cos(2\pi) = 32 \\ &\vdots \end{aligned}$$

Therefore, the Taylor series centered at $x = \pi$ is

$$\begin{aligned} 0 + \frac{2}{1!}(x-\pi) + 0 - \frac{8}{3!}(x-\pi)^3 + 0 + \frac{32}{5!}(x-\pi)^5 + \ldots &= \frac{2}{1!}(x-\pi) - \frac{8}{3!}(x-\pi)^3 + \frac{32}{5!}(x-\pi)^5 - \ldots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1}. \end{aligned}$$