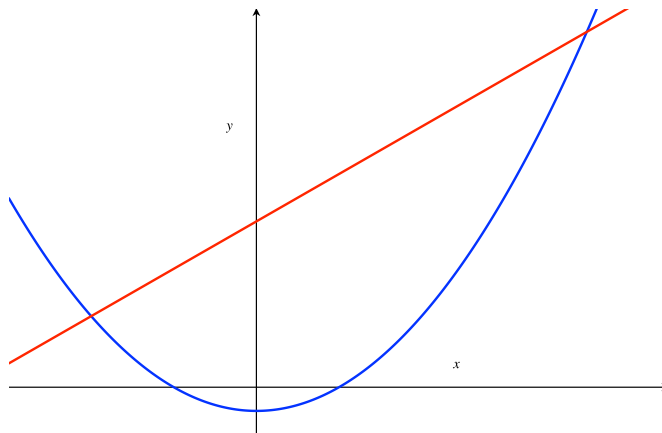


Math 2260 Exam #1 Practice Problem Solutions

1. What is the area bounded by the curves $y = x^2 - 1$ and $y = 2x + 7$?



Answer: As we can see in the figure, the line $y = 2x + 7$ lies above the parabola $y = x^2 - 1$ in the region we care about. Also, the points of intersection occur when $2x + 7 = x^2 - 1$ or, equivalently, when

$$0 = x^2 - 2x - 8 = (x - 4)(x + 2),$$

so the curves intersect when $x = 4$ and $x = -2$. Therefore, integrating top minus bottom over this region should yield the area between the curves:

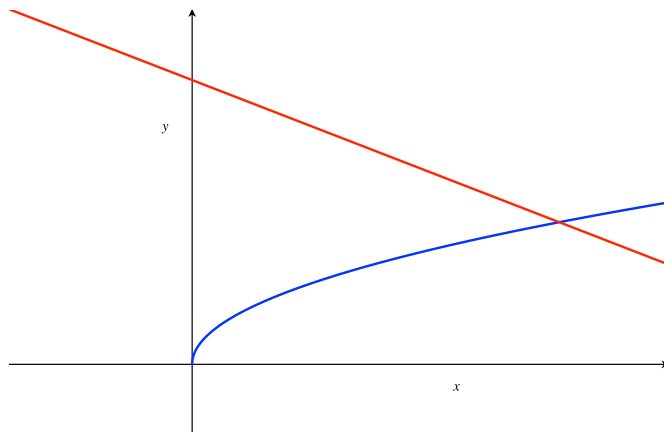
$$\begin{aligned} \int_{-2}^4 [(2x + 7) - (x^2 - 1)] dx &= \int_{-2}^4 (2x + 8 - x^2) dx \\ &= \left[x^2 + 8x - \frac{x^3}{3} \right]_{-2}^4 \\ &= \left(16 + 32 - \frac{64}{3} \right) - \left(4 - 16 + \frac{16}{3} \right) \\ &= 48 - \frac{64}{3} + 12 - \frac{8}{3} \\ &= 60 - \frac{72}{3} \\ &= 60 - 24 \\ &= 36. \end{aligned}$$

So the area between the curves is $\frac{100}{3}$.

2. What is the volume of the solid obtained by rotating the region bounded by the graphs of $y = \sqrt{x}$, $y = 2 - x$ and $y = 0$ around the x -axis?

Answer: As we see in the figure, the line $y = 2 - x$ lies above the curve $y = \sqrt{x}$ in the region we care about. We're revolving around the x -axis, so washers will be vertical and cylindrical shells will have horizontal sides. We would need to split the computation up into two integrals if we wanted to use the shell method, so we'll use the washer method. The area of a cross section will be

$$A(x) = \pi(2 - x)^2 - \pi(\sqrt{x})^2 = \pi(4 - 4x + x^2) - \pi x = \pi(4 - 5x + x^2).$$



Now, the region runs from $x = 0$ until the curves cross, which happens when $2 - x = \sqrt{x}$, which is to say when $x = 1$. Therefore, the volume of the solid will be

$$\begin{aligned} \int_0^1 A(x) dx &= \pi \int_0^1 (4 - 5x + x^2) dx \\ &= \pi \left[4x - \frac{5}{2}x^2 + \frac{x^3}{3} \right]_0^1 \\ &= \pi \left[4 - \frac{5}{2} + \frac{1}{3} \right] \\ &= \frac{11\pi}{6}. \end{aligned}$$

Therefore, the volume of the solid must be $\frac{11\pi}{6}$.

3. What is the volume obtained by revolving the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ around the line $x = 2$?

Answer: I didn't give a picture of the region, but the two curves are a parabola and an upside-down parabola, so we know they must bound a region that looks something like the shadow of a football. Moreover, the curve $y = 4 - x^2$ is clearly above $y = x^2 - 4$ in this region (for example, at $x = 0$). The curves intersect when $4 - x^2 = x^2 - 4$, meaning when

$$8 = 2x^2,$$

so the intersections happen when $x = \pm 2$.

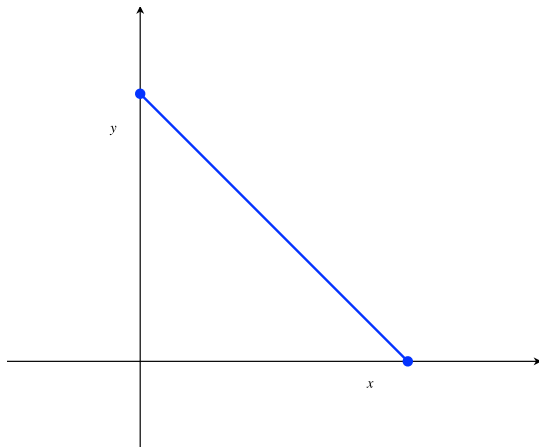
Now, we're revolving around a vertical line, so washers would be horizontal and shells would have vertical sides. The shell is clearly preferable, since the vertical sides will simply run from $y = 4 - x^2$ to $y = x^2 - 4$, whereas for washers the inner and outer sides would both be determined by $y = 4 - x^2$ on the top half of the solid and by $y = x^2 - 4$ on the bottom half of the solid.

Since we're using cylindrical shells and the region runs from $x = -2$ to $x = 2$, the volume of the solid

is

$$\begin{aligned}\int_{-2}^2 2\pi(2-x)((4-x^2)-(x^2-4)) dx &= 2\pi \int_{-2}^2 (2-x)(8-2x^2) dx \\ &= 2\pi \int_{-2}^2 (16-8x-4x^2+2x^3) dx \\ &= 2\pi \left[16x - 4x^2 - \frac{4}{3}x^3 + \frac{x^4}{2} \right]_{-2}^2 \\ &= 2\pi \left[\left(32 - 16 - \frac{32}{3} + 8 \right) - \left(-32 - 16 + \frac{32}{3} + 8 \right) \right] \\ &= 2\pi \left[64 - \frac{64}{3} \right] \\ &= \frac{256\pi}{3}\end{aligned}$$

4. The base of a solid is the triangle in the xy -plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$. The cross-sections of the solid perpendicular to the x -axis are squares. What is the volume of the solid?



Answer: Since we're given the shape of a cross-section perpendicular to the x -axis, the area of the cross-section will change as x changes, so we should integrate with respect to x from $x = 0$ to $x = 1$. Now, each cross section is just a square whose base runs from the blue line in the picture to the x -axis. The equation of the line is $y = 1 - x$, so the length of the base of the square is $(1 - x) - 0 = 1 - x$. Therefore, the area of a cross-section is given by

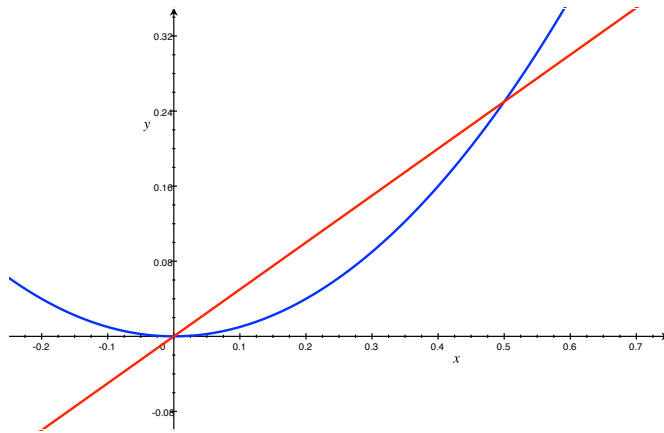
$$A(x) = (1 - x)^2 = 1 - 2x + x^2.$$

Hence, the volume of the solid is

$$\begin{aligned}\int_0^1 A(x) dx &= \int_0^1 (1 - 2x + x^2) dx \\ &= \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\ &= 1 - 1 + \frac{1}{3} \\ &= \frac{1}{3}.\end{aligned}$$

So we see that the volume of the solid is $\frac{1}{3}$.

5. Find the volume of the solid obtained by rotating the area between the graphs of $y = x^2$ and $x = 2y$ around the y -axis.



Answer: First, notice that the two curves intersect when $x^2 = x/2$, which means either $x = 0$ or $x = 1/2$.

Now, we're revolving around the y -axis, which is a vertical line, so washers would be horizontal and cylindrical shells would have vertical sides. We can actually use either method to find the volume of the solid.

To use cylindrical shells, notice that the sides of the cylinder will run from the red line to the blue curve, and so the shells will have height $\frac{x}{2} - x^2$. Also, for a given x , the cylinder at x will have radius $x - 0 = x$, so the volume of the solid is

$$\begin{aligned} \int_0^{1/2} 2\pi x \left(\frac{x}{2} - x^2 \right) dx &= 2\pi \int_0^{1/2} \left(\frac{x^2}{2} - x^3 \right) dx \\ &= 2\pi \left[\frac{x^3}{6} - \frac{x^4}{4} \right]_0^{1/2} \\ &= 2\pi \left[\frac{1}{48} - \frac{1}{64} \right] \\ &= 2\pi \frac{1}{192} \\ &= \frac{\pi}{96}. \end{aligned}$$

If we instead used washers, notice that the washers are horizontal, so their area changes as we change y . Therefore, the area of the washer should be a function of y , meaning we should express both of our functions as functions of y . Then the red line is the graph of $x = 2y$, and the blue curve is the graph of $x = \sqrt{y}$. Now, the outer radius of each washer is the distance from the blue curve to the y -axis, which is $\sqrt{y} - 0 = \sqrt{y}$, while the inner radius is the distance from the red line to the y -axis, which is $2y - 0 = 2y$. Therefore, the area of a cross-sectional washer will be

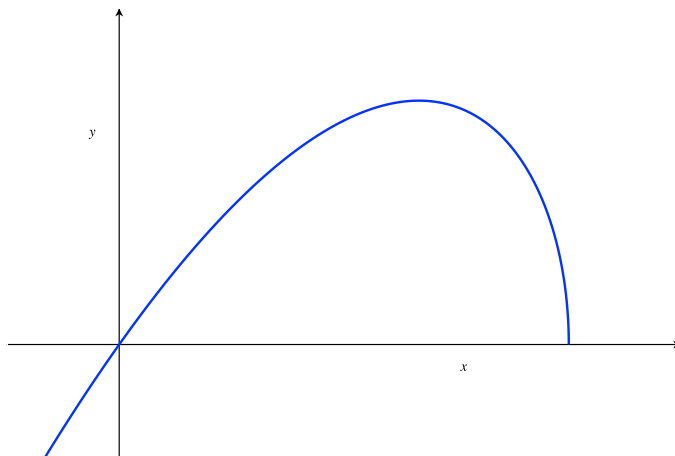
$$A(y) = \pi (\sqrt{y})^2 - \pi (2y)^2 = \pi y - 4\pi y^2 = \pi (y - 4y^2).$$

Finally, we need to determine the limits of integration. These will be given by the points of intersection, which are determined by $2y = \sqrt{y}$, which means $y = 0$ or $y = 1/4$. Therefore, the volume of the solid

is given by

$$\begin{aligned}\int_0^{1/4} A(y) dy &= \int_0^{1/4} \pi (y - 4y^2) dy \\ &= \pi \left[\frac{y^2}{2} - \frac{4}{3}y^3 \right]_0^{1/4} \\ &= \pi \left(\frac{1}{2 \cdot 16} - \frac{1}{3 \cdot 16} \right) \\ &= \pi \left(\frac{3}{6 \cdot 16} - \frac{2}{6 \cdot 16} \right) \\ &= \frac{\pi}{6 \cdot 16} \\ &= \frac{\pi}{96}.\end{aligned}$$

6. Find the volume of the solid obtained by rotating the region between the graphs of $y = x\sqrt{2-x}$ and $y = 0$ around the x -axis.



Answer: We're rotating around the x -axis, so washers would be vertical and cylindrical shells would be horizontal. There's clearly a problem with using cylindrical shells, as their heights would be given by the distance from the curve to itself, which is tricky to get a handle on.

Instead, let's use washers. Since the washers are vertical, their areas change as the variable x -changes, so we should express the cross-sectional area as a function of x . Since the "washer" is actually just a disk of radius $x\sqrt{2-x}$, we know that the cross-sectional area is

$$A(x) = \pi (x\sqrt{2-x})^2 = \pi (x^2(2-x)) = \pi (2x^2 - x^3).$$

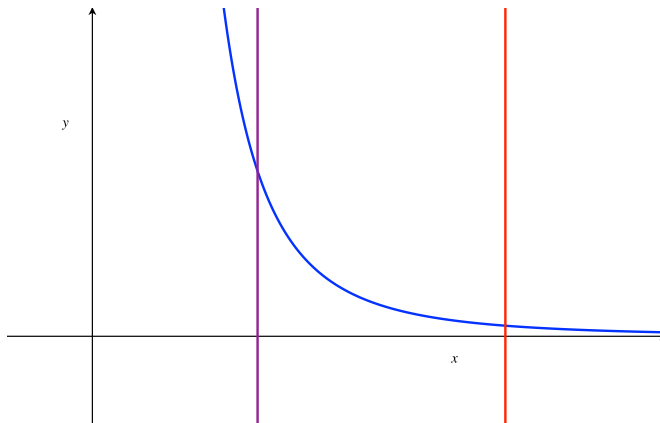
Now, the blue curve crosses the x -axis when $x\sqrt{2-x} = 0$, which happens when $x = 0$ and $x = 2$, so

these should be our limits of integration. Hence, the volume of the solid is

$$\begin{aligned} \int_0^2 A(x) dx &= \int_0^2 \pi (2x^2 - x^3) dx \\ &= \pi \left[\frac{2}{3}x^3 - \frac{x^4}{4} \right]_0^2 \\ &= \pi \left[\frac{16}{3} - \frac{16}{4} \right] \\ &= \frac{4\pi}{3}. \end{aligned}$$

7. Let $V(b)$ be the volume obtained by rotating the area between the x -axis and the graph of $y = \frac{1}{x^3}$ from $x = 1$ to $x = b$ around the x -axis. What is $V(b)$? Can you say anything about what happens to $V(b)$ as b goes to ∞ ?

Answer: I didn't give a picture, but you should be able to guess that the picture is something like this:



Here the purple line is $x = 1$ and the red line is $x = b$. We're revolving around the x -axis, so washers would be vertical and cylindrical shells would have horizontal sides. Figuring out the height of a shell is clearly going to be messy, so let's use washers. Since the washers are vertical, their area changes as x changes, so we should express the area of the washer as a function of x . Since the washer is actually a disk with radius $\frac{1}{x^3} - 0 = \frac{1}{x^3}$, we see that the cross-sectional area is

$$A(x) = \pi \left(\frac{1}{x^3} \right)^2 = \pi \cdot \frac{1}{x^6}.$$

Since the region runs from $x = 1$ to $x = b$, those are our limits of integration, and the volume of the

solid is

$$\begin{aligned}V(b) &= \int_1^b A(x) dx \\&= \int_1^b \pi \frac{1}{x^6} dx \\&= \pi \left[\frac{-1}{5x^5} \right]_1^b \\&= \pi \left[\frac{-1}{5b^5} + \frac{1}{5} \right] \\&= \pi \left(\frac{1}{5} - \frac{1}{5b^5} \right).\end{aligned}$$

As b goes to ∞ , the term $\frac{1}{5b^5}$ goes to zero rather quickly, so the function $V(b)$ goes to $\frac{\pi}{5}$ as $b \rightarrow \infty$.

8. Write down an integral which will compute the length of the part of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$. Don't worry about evaluating this integral.

Answer: I plan to use the arc length integral, which says that the length of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

so I need to figure out $\frac{dy}{dx}$. Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (\ln(\cos x)) \\&= \frac{1}{\cos x} \frac{d}{dx} (\cos x) \\&= \frac{1}{\cos x} (-\sin x) \\&= \frac{-\sin x}{\cos x}.\end{aligned}$$

Therefore, the length of the curve from $x = 0$ to $x = \pi/4$ is given by the integral

$$\int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2} dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/4} \sec x dx.$$

At this point, we haven't yet learned how to find the antiderivative of $\sec x$, so this is as far as we can go.

9. Calculate the surface area of the surface obtained by revolving the curve $y = \frac{x^3}{3}$ around the x -axis for $1 \leq x \leq 2$.

I plan to use the fact that the surface area of a surface given by revolving the graph of $y = f(x)$ around the x -axis from $x = a$ to $x = b$ is given by

$$\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

Therefore, it's important to know $f'(x)$ (or, saying the same thing, $\frac{dy}{dx}$). But of course $f'(x) = x^2$, so the surface area between $x = 1$ and $x = 2$ will be

$$\int_1^2 2\pi \frac{x^3}{3} \sqrt{1 + (x^2)^2} dx = \int_1^2 2\pi \frac{x^3}{3} \sqrt{1 + x^4} dx.$$

Let $u = 1 + x^4$. Then $du = 4x^3 dx$ and we can write the above integral as

$$\begin{aligned} \frac{2\pi}{3} \cdot \frac{1}{4} \int_1^2 4x^3 \sqrt{1 + x^4} dx &= \frac{\pi}{6} \int_2^{17} \sqrt{u} du \\ &= \frac{\pi}{6} \left[\frac{2}{3} u^{3/2} \right]_2^{17} \\ &= \frac{\pi}{3} \left[\frac{17\sqrt{17}}{3} - \frac{2\sqrt{2}}{3} \right] \\ &= \frac{\pi}{9} (17\sqrt{17} - 2\sqrt{2}) \end{aligned}$$

10. Calculate the surface area of the surface obtained by revolving the curve $y = \sqrt{9 - x^2}$ around the x -axis for $1 \leq x \leq 3$.

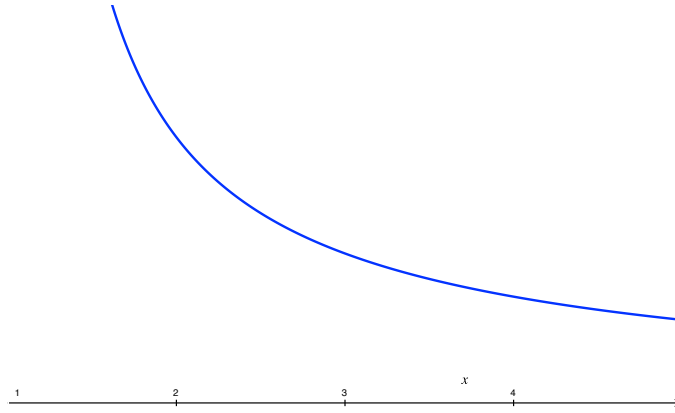
Answer: Again, I intend to use the surface area integral, so I need to know $\frac{dy}{dx}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (\sqrt{9 - x^2}) \\ &= \frac{d}{dx} [(9 - x^2)^{1/2}] \\ &= \frac{1}{2} (9 - x^2)^{-1/2} \cdot \frac{d}{dx} (9 - x^2) \\ &= \frac{1}{2\sqrt{9 - x^2}} \cdot (-2x) \\ &= \frac{-x}{\sqrt{9 - x^2}}. \end{aligned}$$

Therefore, the surface area of the surface is given by

$$\begin{aligned} \int_1^3 2\pi \sqrt{9 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}} \right)^2} dx &= 2\pi \int_1^3 \sqrt{9 - x^2} \sqrt{1 + \frac{x^2}{9 - x^2}} dx \\ &= 2\pi \int_1^3 \sqrt{9 - x^2} \sqrt{\frac{9 - x^2}{9 - x^2} + \frac{x^2}{9 - x^2}} dx \\ &= 2\pi \int_1^3 \sqrt{9 - x^2} \sqrt{\frac{9}{9 - x^2}} dx \\ &= 2\pi \int_1^3 \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}} dx \\ &= 2\pi \int_1^3 3 dx \\ &= 2\pi [3x]_1^3 \\ &= 2\pi(9 - 3) \\ &= 12\pi \end{aligned}$$

11. Find the area between $y = \frac{x}{x^2-1}$ and the x -axis for $2 \leq x \leq 4$.



Answer: Clearly, we just need to integrate $\frac{x}{x^2-1} - 0 = \frac{x}{x^2-1}$ from $x = 2$ to $x = 4$ to find this area; in symbols:

$$\int_2^4 \frac{x}{x^2-1} dx.$$

To compute this integral, I will use u -substitution. Let $u = x^2 - 1$. Then $du = 2x dx$, and so the above integral is equal to

$$\begin{aligned} \frac{1}{2} \int_2^4 \frac{2x}{x^2-1} &= \frac{1}{2} \int_3^{15} \frac{du}{u} \\ &= \frac{1}{2} [\ln u]_3^{15} \\ &= \frac{1}{2} [\ln 15 - \ln 3] \\ &= \frac{1}{2} \ln 5 \\ &= \ln \sqrt{5}. \end{aligned}$$

12. What is the length of the curve $y = \sqrt{x} - \frac{x^{3/2}}{3}$ between $x = 0$ and $x = 2$?

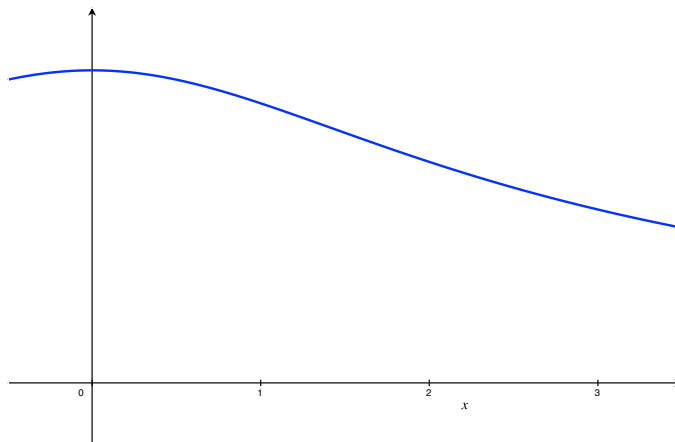
Answer: Once again, I plan to use the arc length integral, so I need to know $\frac{dy}{dx}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{x} - \frac{x^{3/2}}{3} \right) \\ &= \frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}. \end{aligned}$$

Therefore, the length of the curve from $x = 0$ to $x = 2$ is given by

$$\begin{aligned}
 \int_0^2 \sqrt{1 + \left(\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2} dx &= \int_0^2 \sqrt{1 + \left(\frac{1}{4x} - \frac{1}{2} + \frac{x}{4}\right)} dx \\
 &= \int_0^2 \sqrt{\frac{1}{4x} + \frac{1}{2} + \frac{x}{4}} dx \\
 &= \int_0^2 \sqrt{\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right)^2} dx \\
 &= \int_0^2 \left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right) dx \\
 &= \left[\sqrt{x} + \frac{x^{3/2}}{3}\right]_0^2 \\
 &= \sqrt{2} + \frac{2\sqrt{2}}{3} \\
 &= \frac{5\sqrt{2}}{3}.
 \end{aligned}$$

13. What is the volume obtained by rotating the region between $y = \frac{1}{\sqrt{x^2+4}}$ and the x -axis for $0 \leq x \leq 3$ around the y -axis?



Answer: Since the axis of rotation is vertical, washers will be horizontal and cylindrical shells will have vertical sides. Clearly, finding the area of washers will be a problem, since some will have outer edge given by the line $x = 3$ and some will have outer edge given by the curve. Instead, I will use cylindrical shells. Since the shells change as x changes, I will be expressing everything in terms of x .

For a given x , the corresponding shell has radius x and height given by the distance from the curve to the x -axis, which is just $\frac{1}{\sqrt{x^2+4}} - 0 = \frac{1}{\sqrt{x^2+4}}$. Therefore, the volume of the solid will be

$$\int_0^3 2\pi x \left(\frac{1}{\sqrt{x^2+4}}\right) dx = \pi \int_0^3 \frac{2x dx}{\sqrt{x^2+4}}.$$

Let $u = x^2 + 4$. Then $du = 2x dx$, and so the above integral is equal to

$$\begin{aligned}\pi \int_4^{13} \frac{du}{\sqrt{u}} &= \pi [2\sqrt{u}]_4^{13} \\ &= \pi (2\sqrt{13} - 2\sqrt{4}) \\ &= 2\pi\sqrt{13} - 4\pi.\end{aligned}$$